MAT 111
(DLC)

by

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Contents

Lecture

General Introduction and Objectives

We shall study topics which are usually referred to as Pre-Calculus Mathematics. We start with a study of Polynomials and Polynomial equations and inequalities, followed by Rational Functions. After studying the principle of mathematical induction, we go on to study Permutations and Combinations, the Binomial Theorem, Sequences and Series. We shall then study the algebra and geometry of Complex Numbers.

We shall then study the algebra of matrices, together with their properties of determinant and rank, and apply them to the solution of any system of m linear equations in n unknowns, as well as transformations of the plane.

We shall also conclude with an introduction to the theory of Sets.

The objectives of the course are as follows. The reader should be able to:

- manipulate polynomials, polynomial equations and inequalities, as well as rational functions,
- apply the principle of mathematical induction,
- solve problems involving permutations and combinations, the binomial theorem, sequences and series,
- solve problems on the algebra and geometry of complex numbers,
- state and apply the properties of the algebra of matrices, their determinants and ranks,
- apply matrices to solve any system of m linear equations in n unknowns, and
- solve problems on translations in the plane, and binary operations on sets.

LECTURE ONE

Polynomials and Polynomial Equations

Introduction

We shall study how to perform the basic operations of addition, subtraction, multiplication and division on polynomials. We shall then study how to solve different types of polynomial equations of degree ≤ 2.

We shall conclude by applying the remainder and factor theorems to the solution of polynomial equations of degree > 2.

Objectives

The reader should be able to

- perform the basic operations on polynomials,
- solve different types of polynomial equations of degree ≤ 2, and
- apply the remainder and factor theorems to the solution of polynomial equations of degree > 2.

Pre-Test

1(a) Find the quotient (Q) and the remainder (R):

$$(x^3 - 5x^2 + 7x - 2) \div (3x - 2)$$

(b) If $f(x) = 3x^2 - 2x + 1$, evaluate $f(-5)$.

2. Solve for x:

   (a) $\frac{1}{5}(x-3) - \frac{1}{6}(x+2) = \frac{1}{10}(2-x) - 1$

   (b) $\frac{3x+5}{x+10} - \frac{2x+1}{x+8} = 1$

   (c) $\sqrt{x} - \sqrt{x-2} = 1$

   (d) $4^{x+1} = 2^{3x-1}$

   (e) $\log_2(2x-1) - \log_2(2-x) = 3$

3. Solve simultaneously for x and y:

$$3^{2x+y} = 3, \quad 10^{3x+y} = 100$$

4. Solve for x:

$$3^{2x} - 28(3^x) + 27 = 0$$

5. Solve the simultaneous equations:

$$2x + y = 4, \quad x^2 + xy = -12$$

6. If α and β are the roots of the equation $3x^2 - 8x - 4 = 0$, find the value of (a) $\alpha^2 + \beta^2$, (b) $(\alpha - \beta)^2$, (c) $\alpha^3 + \beta^3$, (d) $\alpha^3 - \beta^3$.

7. Find the value of the discriminant, and hence determine the nature of the roots of the equation

$$2x^2 - 5x + 7 = 0.$$

8. Use the Remainder Theorem to find the remainder when $6x^4 + 2x^3 - 1$ is divided by $2x + 1$.

9. If a polynomial $f(x)$ is divided by $(x-2)$, the remainder is 1 and when $f(x)$ is divided by $(x+3)$, the remainder is 5. What is the remainder, when $f(x)$ is divided by $(x-2)(x+3)$?

10. Solve the equation:
$$x^3 + x^2 - 4x - 4 = 0.$$

Addition, Subtraction and Multiplication of Polynomials
Example 1. Simplify

$$\begin{aligned} 3(2x^3 - 4x - 1) + 2(x^3 - x + 4) &= 6x^3 - 12x - 3 + 2x^3 - 2x + 8 \\ &= 8x^3 - 14x + 5 \end{aligned}$$

Example 2. Simplify:

$$\begin{aligned} & 3(x^3 + 3x^2 - 5) - 2(2x^3 - x^2 + 2x - 4) \\ = \; & 3x^3 + 9x^2 - 15 - 4x^3 + 2x^2 - 4x + 8 \\ = \; & -x^3 + 11x^2 - 4x - 7 \end{aligned}$$

Example 3. Simplify:

$$\begin{aligned} & (x^2 - 2x + 4)(x^3 - x + 4) \\ = \; & x^3(x^3 - x + 4) - 2x(x^3 - x + 4) + 4(x^3 - x + 4) \\ = \; & x^6 - x^4 + 4x^3 - 2x^4 + 2x^2 - 8x + 4x^3 - 4x + 16 \\ = \; & x^6 - 3x^4 + 8x^3 + 2x^2 - 12x + 16 \end{aligned}$$

Example 4
If $f(x) = 2x^3 - 3x + 4$, evaluate $f(-2)$

$$f(-2) = 2(-2)^3 - 3(-2) + 4 = -16 + 6 + 4 = -6$$

Division of Polynomials
In the expression, $f(x) \div g(x)$, $f(x)$ is called the <u>dividend</u> and $g(x)$ is called the <u>divisor</u>.
If the degree of $g(x)$ is less than or equal to the degree of $f(x)$, then we can use a division algorithm, similar to the method of long division in numbers, and obtain a quotient $q(x)$ and a remainder $f(x)$ of degree less than degree of $g(x)$, so that
$$f(x) = g(x) \cdot g(x) + r(x)$$

or

$$\frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}.$$

Example 5.

Obtain the quotient and remainder when $x^3 + 2x^2 - 5x + 1$ is divided by $x + 1$.

$$\begin{array}{r|l}
 & x^2 + x - 6 \\
\hline
\underline{x+1} & x^3 + 2x^2 - 5x + 1 \\
 & \underline{-(x^3 + x^2)} \\
 & x^2 - 5x + 1 \\
 & \quad \underline{-(x^2 + x)} \\
 & \quad -6x + 1 \\
 & \quad \underline{-(-6x - 6)} \\
 & \qquad 7
\end{array}$$

Quotient $= x^2 + x - 6$, Remainder $= 7$

$$x^3 + 2x^2 - 5x + 1 = (x+1)(x^2 + x - 6) + 7$$

or

$$\frac{x^3 + 2x^2 - 5x + 1}{x+1} = x^2 + x - 6 + \frac{7}{x+1}$$

Practice Exercise 1A

1. Simplify
   (a) $2(1 + 2x + 3x^2 + 4x^3) - 3(1 - 2x + x^2)$
   (b) $(2x^2 - 4x - 1)(x^2 - x + 4)$
2. Find the quotient and the remainder
   (a) $(3x^2 - 4x + 5) \div (x - 1)$
   (b) $(x^3 - 2x^2 + 3x - 6) \div (x + 3)$
   (c) $(x^3 + 6x^2 - 7x + 4) \div (x^2 + 1)$

3. If $f(x) = x^3 + x^2 - 9x + 9$, evaluate
   (a) $f(0)$, (b) $f(-\frac{1}{2})$, (c) $f(-x)$, (d) $f(x+h)$

Polynomial equations
We shall consider the following types:

(i) Linear equations in one variable;

(ii) Simultaneous linear equations in two variables;

(iii) Quadratic equations.

Example 1. Solve for x:

$$\frac{3}{5}(x+1) - \frac{2}{3}(2x-5) = \frac{1}{30}(2+7x)$$

Multiply by the LCM of 5, 3 and 30, which is 30:

$$18(x+1) - 20(2x-5) = 2 + 7x$$

$$-29x = -116, \quad x = 4$$

Example 2. Solve for x and y

$$5x + 3y = 1 \tag{1}$$

$$2x - 7y = 3 \tag{2}$$

$$\begin{array}{rcl} (1) \times 2 & : & 10x + 6y = 2 \\ (2) \times 5 & : & 10x - 35y = 15 \\ \hline \text{Subtract} & : & 41y = -13, \; y = -\dfrac{13}{41} \end{array}$$

From (2): $x = \frac{1}{2}(3+7y) = \frac{1}{2}\left(3 - \frac{91}{41}\right) = \frac{16}{41}$

<u>Check</u>: (1): L.H.S. $= 5\left(\frac{16}{41}\right) - 3\left(\frac{13}{41}\right) = 1 =$ R.H.S.

Example 3. Solve for x, by factorization:

$$4x^2 - 4x - 15 = 0.$$

In the general quadratic equation:

$$ax^2 + bx + c = 0$$

$a = 4$, $b = -4$, $c = -15$.
If the discriminant, $D = b^2 - 4ac$, is a perfect square, then we can factorize with integral coefficients:

$$D = (-4)^2 - 4(4)(-15) = 256 = 16^2.$$

Factorize: $4(-15) = (-10)(+6)$ so that

$$\begin{array}{l} -4x = -10x + 6x \\ 4x^2 - 10x + 6x - 15 = 0 \\ 2x(2x-5) + 3(2x-5) = 0 \\ (2x+3)(2x-5) = 0 \\ x = -\dfrac{3}{2} \text{ or } \dfrac{5}{2} \end{array}$$

Example 4. Solve for x, by completing the square:

$$x^2 - 2x - 20 = 0$$

$a = 1$, $b = -2$, $c = -20$.
$D = b^2 - 4ac = (-2)^2 - 4(1)(-20) = 84$ is not a perfect square.

$$\begin{aligned} x^2 - 2x &= 20 \\ x^2 - 2x + (-1)^2 &= 20 + (-1)^2 \\ (x-1)^2 &= 21 \\ x &= 1 \pm \sqrt{21} \end{aligned}$$

Example 5. Solve for x, using the quadratic formula:

$$3x^2 - 11x - 6 = 0.$$

The solutions of $ax^2 + bx + c = 0$ are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 3$, $b = -11$, $c = -6$.

$$\begin{aligned} x &= \frac{11 \pm \sqrt{(-11)^2 - 4(3)(-6)}}{6} \\ &= \frac{11 \pm 193}{6} \end{aligned}$$

Example 6. Solve for x:

$$\begin{array}{l} 2^{2x+1} + 2^{3-2x} = 17 \\ 2^1 \cdot 2^{2x} + 2^3 \cdot 2^{-2x} = 17 \end{array}$$

Put $y = 2^{2x}$, $2^{-2x} = (2^{2x})^{-1} = y^{-1}$

$$\begin{array}{rl} & 2y + 8y^{-1} = 17 \\ Xy: & 2y^2 - 17y + 8 = 0 \\ & a = 2,\ b = -17,\ c = 8 \\ & D = b^2 - 4ac = (-17)^2 - 4(2)(8) = 225 = 15^2 \end{array}$$

Factorize since $2(8) = (-16)(-1)$:

$$\begin{aligned} 2y^2 - 16y - y + 8 &= 0 \\ 2y(y-8) - (y-8) &= 0 \\ (2y-1)(y-8) &= 0 \end{aligned}$$

$y = \frac{1}{2}$ or 8
i.e. $2^{2x} = 2^{-1}$ or 2^3
$2x = -1$ or 3, $x = -\dfrac{1}{2}$ or $\dfrac{3}{2}$.

Example 7. Solve for x and y:

$$x + y = 5 \tag{1}$$

$$x^2 - 2y^2 = 1 \tag{2}$$

From (1): $y = 5 - x$
substitute for y in (2)

$$\begin{array}{l} x^2 - 2(5-x)^2 = 1 \\ x^2 - 20x + 51 = 0 \\ (x-17)(x-3) = 0,\ x = 3 \text{ or } 17 \end{array}$$

From (1), $y = 2$ or -12.
Solutions are $x = 3$, $y = 2$ or $x = 17$, $y = -12$.

Example 8. Find the nature of the roots of the general quadratic equation

$$ax^2 + bx + c = 0$$

From the quadratic formula

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2} \\ &= \frac{-b \pm \sqrt{D}}{2} \end{aligned}$$

where $D = b^2 - 4ac$ is the discriminant.

(a) If $D > 0$, there are 2 unequal real roots.

(b) If $D = 0$, there are 2 equal real roots, i.e., only one real root.

(c) If $D < 0$, there are no real roots because the square root of a negative number is not a real number.

Example 9

Find the sum and the product of roots of the general equation

$$ax^2 + bx + c = 0.$$

Let the roots be α and β. Then

$$\alpha = \frac{-b}{2a} + \frac{\sqrt{D}}{2a}, \ \beta = \frac{-b}{2a} - \frac{\sqrt{D}}{2a}, \ D = b^2 - 4ac.$$

$$\begin{aligned} \text{Sum of roots} &= \alpha + \beta = \frac{-b}{a} \\ \text{Product of roots} &= \alpha\beta = \left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{D}}{2a}\right)^2 \\ &= \frac{b^2}{4a^2} - \frac{D}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} \\ &= \frac{4ac}{4a^2} = \frac{c}{a} \end{aligned}$$

Example 10.
If α and β are the roots of the equation

$$3x^2 + 8x - 4 = 0$$

find the value of $\alpha^2 + \beta^2$

$$\begin{aligned}
a &= 3,\ b = 8,\ c = -4 \\
\alpha + \beta &= \frac{-b}{a} = -\frac{8}{3},\ \alpha\beta = \frac{c}{a} = \frac{-4}{3} \\
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\
&= \left(\frac{-8}{3}\right)^2 - 2\left(\frac{-4}{3}\right) = \frac{64}{9} + \frac{8}{3} = \frac{88}{9}.
\end{aligned}$$

Practice Exercise 1B

1. Solve for x:
   (a) $\frac{1}{6}(3x-1)(2x+3) - \frac{1}{4}(2x+5)^2 = \frac{19}{4}$
   (b) $4\sqrt{x+1} = 3\sqrt{x+2}$
   (c) $(2x-5)(3x+4) = (1-2x)(5-3x)$
   (d) $3^{2x-1} = \frac{1}{81}$
   (e) $\log_2(2x^2 - 3x + 2) - 2\log_2(x+5) = 1$.
2. Solve for x and y:
   (a) $8x - 5y = 25$, $9x - 7y = 24$
   (b) $\log_2 x - \log_2 y = 2$, $2\log_2 x + 3\log_2 y = 19$
3. Solve for x
   (a) $7x^2 + 8x - 12 = 0$, by factorization
   (b) $2x^2 - 7x - 3 = 0$, by completing the square
   (c) $4x^2 + 5x - 2 = 0$, by the quadratic formula

(d) $2^{2x} + 2(2^x) - 8 = 0$

4. Solve for x and y

   (a) $x^2 + y^2 = 25$, $x + 3y = 5$

   (b) $x^2 + y - 8 = 0$, $y + 5x - 2 = 0$

5. If the roots of $2x^2 + 6x - 5 = 0$ are α and β, find the values of (a) $\alpha^2 + \beta^2$, (b) $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$, (c) $(\alpha - \beta)^2$

6. Find the value of the discriminant, and hence determine the nature of the roots of

   (a) $4x^2 + 4x + 1 = 0$

   (b) $5x^2 - x - 4 = 0$

   (c) $5x^2 + 6x + 5 = 0$

Remainder and Factor Theorems

Example 1. Find the remainder when a polynomial $f(x)$ of degree ≥ 1 is divided by $(ax + b)$.

By the division algorithm, the remainder must be of degree 0, since the divisor $(ax + b)$ is of degree 1. Hence the remainder is a constant R, say.

$$f(x) = (ax + b) \cdot q(x) + R$$

Put $x = -\dfrac{b}{a}$, obtained by solving $ax + b = 0$,

$$\begin{aligned} f\left(-\frac{b}{a}\right) &= \left(a\left(-\frac{b}{a}\right) + b\right) q\left(-\frac{b}{a}\right) + R \\ &= (-b + b)q\left(-\frac{b}{a}\right) + R = R \end{aligned}$$

i.e. the remainder $R = f\left(-\dfrac{b}{a}\right)$.

Remark. Example 1 is called the Remainder Theorem. It is used to find the remainder without performing the long division. The remainder theorem is true only when the divisor is a LINEAR function.

Example 2. Find $f\left(-\dfrac{b}{a}\right)$ if $(ax + b)$ is a factor of a polynomial $f(x)$ of degree ≥ 1.
From Example 1 above, the remainder $R = 0$. Hence

$$f\left(-\frac{b}{a}\right) = 0.$$

Remarks

1. Example 2 is called the Factor Theorem.
2. Also conversely, if $f\left(-\dfrac{b}{a}\right) = 0$, then $(ax + b)$ is a factor of $f(x)$.
3. The factor theorem is used to factorize polynomials of degree > 2.

Example 3.
Find the unknown coefficient if $x - 2$ is a factor of $ax^2 - 12x + 4$.
Put $f(x) = ax^2 - 12x + 4$.
Put $x = 2$. Then by the factor theorem

$$f(2) = 0$$

i.e. $a(2^2) - 12(2) + 4 = 0$
$4a - 20 = 0$, $a = 5$.

Example 4.
Find the value of a if the remainder is 1 when $x^2 - 7x + a$ is divided by $x + 1$.
Put $f(x) = x^2 - 7x + a$, $x + 1 = 0 \Rightarrow x = -1$.
Put $x = -1$. Then by the remainder theorem,

$$f(-1) = 1$$

i.e. $(-1)^2 - 7(-1) + a = 1$, $a = -7$.

Example 5.
Find the values of p and q if $(x-2)$ and $(x+1)$ are factors of x^3+px^2+qx+1.
Put $f(x) = x^3 + px^2 + qx + 1$.
$x - 2 = 0 \Rightarrow x = 2$; $x + 1 = 0 \Rightarrow x = -1$.
Put $x = 2$. Then by the factor theorem, $f(2) = 0$, i.e.

$$2^3 + p(2^2) + q(2) + 1 = 0, \; 4p + 2q = -7 \tag{1}$$

Put $x = -1$. Then by the factor theorem, $f(-1) = 0$, i.e.

$$(-1)^3 + p(-1)^2 + q(-1) + 1 = 0 \Rightarrow p - q = 0 \tag{2}$$

Solve (1) and (2) simultaneously $p = q = -\frac{7}{6}$.

Example 6.
Find the values of p and q if when $px^3 - 11x^2 + qx + 4$ is divided by $(x - 1)$ and $(x - 3)$, the remainders are 0 and 70 respectively.
Put $f(x) = px^3 - 11x^2 + qx + 4$.
$x - 1 = 0 \Rightarrow x = 1$. Then by the remainder theorem, $f(1) = 0$, i.e.

$$p(1^3) - 11(1^2) + q(1) + 4 = 0, \;\; p + q = 7 \tag{1}$$

$x - 3 = 0 \Rightarrow x = 3$. Then by the remainder theorem, $f(3) = 70$, i.e.

$$p(3^3) - 11(3^2) + q(3) + 4 = 70, \;\; 27p + 3q = 165$$

i.e.

$$9p + q = 55 \tag{2}$$

Solve (1) and (2) simultaneously,

$$9p + (7 - p) = 55 \Longrightarrow p = 6$$

$(1) \Rightarrow q = 7 - p = 7 - 6 = 1$.

Example 7
Find a and b if when $x^5 + ax^2 + b$ is divided by $x^2 + 5x + 6$, the remainder is $(x + 1)$.
By the division algorithm,

$$x^5 + ax^2 + b = (x^2 + 5x + 6) \cdot q(x) + x + 1$$

$$x^5 + ax^2 + b = (x + 2)(x + 3) \cdot q(x) + x + 1$$

Put $x = -2$:

$$(-2)^5 + a(-2)^2 + b = 0 - 2 + 1, \;\; 4a + b = 31 \tag{1}$$

Put $x = -3$:

$$(-3)^5 + a(-3)^2 + b = 0 - 3 + 1, \;\; 9a + b = 241 \tag{2}$$

Solve (1) and (2) simultaneously:

$$\begin{array}{l} 5a = 210 \Rightarrow a = 42 \\ (1) \Rightarrow b = 31 - 4a = 31 - 4(42) = -137 \end{array}$$

Example 8.
When a polynomial $p(x)$ is divided by $x - 1$ and $x + 2$, respectively, the remainders are 3 and 6. Find the polynomial that is the remainder when $p(x)$ is divided by $(x - 1)(x + 2)$.
By the division algorithm

$$p(x) = (x - 1)(x + 2) \cdot q(x) + r(x)$$

Since the divisor $(x-1) \cdot (x+2)$ is of degree 2, then $r(x) = ax+b$ is of degree at most 1. Hence

$$p(x) = (x - 1)(x + 2) \cdot q(x) + ax + b.$$

Put $x = 1$, then by the remainder theorem, $p(1) = 3$, i.e.

$$0 + a(1) + b = 3, \quad a + b = 3 \tag{1}$$

Put $x = -2$, then by the remainder theorem, $p(-2) = 6$, i.e.

$$0 + a(-2) + b = 6, \quad -2a + b = 6 \tag{2}$$

Solve (1) and (2) simultaneously:

$$3a = -3, \quad a = -1.$$

From (1), $b = 3 - a = 3 - (-1) = 4$..
Hence, the remainder is $ax + b = -x + 4$.

The following Rational Root Theorem gives rational values worth testing when using the Factor Theorem.

Theorem (Rational Root Theorem)
Let $f(x)$ be a polynomial with integral coefficients and let $\frac{p}{q}$ be a rational number in its lowest terms. Then $x = \frac{p}{q}$ is a root of $f(x) = 0$, (or $qx - p$ is a factor of $f(x)$), if

(i) p is a factor of the constant term of $f(x)$, and

(ii) q is a factor of the leading coefficient of $f(x)$.

Example 9.
Solve the equation

$$x^3 - 5x^2 + 7x - 3 = 0$$

Put $f(x) = x^3 - 5x^2 + 7x - 3$.
Values worth testing are ± 1 and ± 3.
$f(1) = 1^3 - 5(1^2) + 7(1) - 3 = 0$
$\Rightarrow (x - 1)$ is a factor.
$f(3) = 3^3 - 5(3^2) + 7(3) - 3 = 0$
$\Rightarrow (x - 3)$ is also a factor.
$\Rightarrow f(x) = (x - 1)(x - 3)(ax + b)$.
Compare coefficients to find a and b.
<u>Coefficients of x^3</u>: $1 = a$
<u>Constant terms</u>: $-3 = 3b \Rightarrow b = -1$.

Hence $f(x) = (x-1)(x-3)(x-1) = 0$.
$\Rightarrow x = 1, 1$ or 3.

Remark: It is possible to use the division algorithm to continue the solution after getting one factor.

Practice Exercise 1c

1. Use the remainder theorem to find the remainder when
   (a) $2x^3 + 6x - 8$ is divided by $x - 1$
   (b) $2x^3 - 5x - 3$ is divided by $2x - 1$
   (c) $6x^3 + 2x^2 - 1$ is divided by $x + 1$
   (d) $x^3 + x^2 + x + 1$ is divided by $2x + 1$.
2. Use the factor theorem to test whether or not
   (a) $2x - 1$ is a factor of $2x^3 - 3x^2 - 3x + 2$
   (b) $2x + 1$ is a factor of $2x^3 + 6x - 8$.
3. Find the unknown coefficients if
   (a) $(x-2)$ is a factor of $x^3 + 7x^2 + ax - 5$
   (b) $(x+4)$ is a factor of $x^3 + ax^2 + 3x - 4$
   (c) $(x-1)$ and $(x-3)$ are factors of $ax^3 - bx^2 - bx + 9$
   (d) $(x-2)$ and $(x-4)$ are factors of $x^4 + ax^3 + bx + 24$
   (e) when $x^6 + cx + d$ is divided by $(x-2)(x+3)$, the remainder is $(2x+1)$.
   (f) when $x^6 + ax^3 + b$ is divided by $(x^2 - 1)$, the remainder is $(2x+3)$.
4. When a polynomial $f(x)$ is divided by $(x+2)$, the remainder is 3, and when $f(x)$ is divided by $(x-1)$, the remainder is -1. Find the remainder when $f(x)$ is divided by $(x+2)(x-1)$.

5. Solve the equation

(i)	$x^3 - 7x - 6 = 0$	(ii)	$x^3 - 7x + 6 = 0$
(iii)	$x^3 + 2x^2 - 1 = 0$	(iv)	$x^3 - 19x + 30 = 0$
(v)	$x^3 - x^2 + x - 1 = 0$	(vi)	$x^3 - 6x^2 + 11x - 6 = 0$

6. Use the factor theorem and the division algorithm to factorize:
   (i) $x^3 + y^3$, (ii) $x^3 - y^3$, (iii) $x^3 + 8$, (iv) $x^3 - 1$.

Summary
We consider addition, subtraction, multiplication and division of polynomials. We then solve different types of polynomial equations such as linear, simultaneous and quadratic equations. We then conclude by applying the remainder and factor theorems to solve certain polynomial equations of degree > 2

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 1-38.

LECTURE TWO

Inequalities

Introduction

We shall first see that inequalities of numbers can be represented using the interval notations, and geometrically on the real number line.

Next we shall study the properties of inequalities, including those of union, intersection and absolute values. We shall then consider solutions of linear, quadratic and cubic inequalities in one variable.

Finally, we shall study problems of order in the real number system.

Objectives

The reader should be able to

- represent inequalities using the interval notation and geometrically on the real number line;
- apply the properties of inequalities to solve linear inequalities and problems of order in the real number system; and
- use the graphical method and the method of the sign table to solve quadratic and cubic inequalities.

Pre-Test

1. Simplify
   (a) $3 < x < 7$ or $-1 < x \leq 4$
   (b) $-2 < x \leq 5$ and $1 \leq x < 8$
2. Solve for x
   (a) $-5 < 4 - 3x \leq 6$
   (b) $|9 - 5x| = 1$
   (c) $|4 - 3x| \leq 5$
   (d) $|2x + 7| > 4$
   (e) $9 - 5x \leq 4 - 3x \leq 2x + 7$
3. Solve for x
   (a) $(2x - 3)(x - 5) > 0$
   (b) $(2 - x)(x + 4) \leq 0$
4. Solve for x
   (a) $(x - 4)(x + 5)(x + 1) \geq 0$
   (b) $(3 - 2x)(x + 3)(x - 3) < 0$
5. Prove that for all real numbers a, b, c and d:
   (a) $a^2b^2 + c^2d^2 \geq 2abcd$
   (b) $a^4 + b^4 \geq a^3b + ab^3$ if $a > 0$ and $b > 0$

Representation of Inequalities

Inequalities	Interval Notation	Number line
$a < x < b$	(a, b) open interval	.
$a \leq x \leq b$	$[a, b]$ closed interval	
$a \leq x < b$	$[a, b)$	
$a < x \leq b$	$(a, b]$	
$x < b$	$(-\infty, b)$	
$x \leq b$	$(-\infty, b]$	
$x > a$	(a, ∞)	
$x \geq a$	$[a, \infty)$	
$-\infty < x < \infty$	$(-\infty, \infty) = \mathbb{R}$	

Union and Intersection
UNION

$$(a < x < b) \text{ or } (c < x < d) = (a, b) \cup (c, d)$$

Example 1

$$2 < x < 5 \text{ or } 3 < x < 7$$

$$(2 < x < 5) \text{ or } (3 < x < 7) = (2,5) \cup (3,7) = (2,7)$$

INTERSECTION

$$(a < x < b) \ \text{ and } \ (c < x < d) = (a,b) \cap (c,d)$$

Example 2

$$-2 \leq x \leq 5 \ \text{ and } \ 1 < x < 7$$

$$-2 \leq x \leq 5 \ \text{ and } \ 1 < x < 7 = [-2,5] \cap (1,7) = (1,5]$$

Properties of Absolute Values

The modulus or the absolute value of a real number x, denoted by $|x|$ is the positive number which has the same magnitude as x. For example,

$$|5| = 5 = |-5|$$

1. $|-x| = |x|$, $|xy| = |x| \cdot |y|$, $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$
2. $|x| \geq 0$, $|x| \geq x$, $|x| \geq -x$ for all real numbers x
   $|x| = 0$ if and only if $x = 0$.
3. $x > 0 \Rightarrow |x| = x$
   $x < 0 \Rightarrow |x| = -x$

4. $|x| = a \Rightarrow (x = a \text{ or } x = -a)$

5. $|x| < a$ means $-a < x < a$

6. $|x| > a$ means $(x > a \text{ or } x < -a)$

7. $|x + y| \leq |x| + |y|$, (Triangle inequality)

Properties of Inequalities

Let a, b, c and d be any real numbers.

1. $a < b$ and $b < c \Rightarrow a < c$ (Transitive law)

2. $a < b \Rightarrow a + c < b + c,\ a - c < b - c$
   $a > b \Rightarrow a + c > b + c,\ a - c > b - c$
   (Adding or Subtracting the same number to both sides does not change the inequality sign).

3. If $c > 0$,
$$a < b \Rightarrow ac < bc,\ \frac{a}{c} < \frac{b}{c}$$
$$a > b \Rightarrow ac > bc,\ \frac{a}{c} > \frac{b}{c}$$
   (Multiplying or Dividing both sides by the same positive number does not change the inequality sign).

4. If $c < 0$,
$$a < b \Rightarrow ac > bc,\ \frac{a}{c} > \frac{b}{c}$$
$$a > b \Rightarrow ac < bc,\ \frac{a}{c} < \frac{b}{c}$$
   (Inequality signs are reversed when we multiply or divide both sides by the same NEGATIVE number)
   e.g.

$$2 < 5 \text{ but } -2 > -5$$

5. If $c \neq 0$, then $c^2 > 0$.

6. $ab > 0$ or $\dfrac{a}{b} > 0$
   $\Rightarrow (a > 0$ and $b > 0)$ or $(a < 0$ and $b < 0)$ since $+ \times + = +$, $- \times - = +$

7. $ab < 0$ or $\dfrac{a}{c} < 0$
   $\Rightarrow$ either $(a > 0$ and $b < 0)$ or $(a < 0$ and $b > 0)$ since $+ \times - = -$, $- \times + = -$

8. $(a < b)$ and $(c < d) \Rightarrow (a + c < b + d)$
   $(a > b)$ and $(c > d) \Rightarrow (a + c > b + d)$

9. If a, b, c and d are all positive
   $(a > b)$ and $(c > d) \Rightarrow (ac > bc)$
   $(a < b)$ and $(c < d) \Rightarrow (ac < bd)$.
   In particular, if a and b are positive

$$a > b \Rightarrow a^2 > b^2$$

$$a < b \Rightarrow a^2 < b^2$$

10. $c > 0 \Rightarrow \dfrac{1}{c} > 0$, $c < 0 \Rightarrow \dfrac{1}{c} < 0$

Linear Inequalities
Example 3. Solve for x:

$$\frac{3x}{4} + \frac{1-x}{3} > 3x - 5$$

Multiply by 12, which is the LCM of 4 and 3. Since $12 > 0$, the inequality sign remains the same.

$$\begin{aligned} 3(3x) + 4(1-x) &> 12(3x-5) \\ 9x + 4 - 4x &> 36x - 60 \\ 5x - 36x &> -60 - 4 \\ -31x &> -64 \end{aligned}$$

Divide by $-31 < 0$, and the inequality sign is reversal:

$$x < \frac{-64}{-31}, \text{ i.e. } x < \frac{64}{31} = \left(-\infty, \frac{64}{31}\right)$$

Example 4
Solve for x

$$3 < 3 - 4x < 7$$

subtract 3: $0 < -4x < 4$.
Divide by $-4 < 0$ and change the inequality signs

$$(-1 < x < 0) = (-1, 0).$$

Example 5
Solve for x

$$|2x - 5| = 3$$

$$\begin{aligned} 2x - 5 &= 3 \text{ or } 2x - 5 = -3 \\ 2x &= 8 \text{ or } 2x = 2 \\ x &= 4 \text{ or } 1. \end{aligned}$$

Example 6
Solve for x

$$\begin{aligned} |1 - 4x| &\leq 5 \\ -5 \leq 1 - 4x &\leq 5 \end{aligned}$$

<u>Subtract 1:</u> $-6 \leq -4x \leq 4$.
Divide by -4 and reverse the inequality signs

$$\left(-1 \leq x \leq \frac{3}{2}\right) = \left[-1, \frac{3}{2}\right].$$

Example 7: Solve for x:

$$|3x + 2| > 5$$

$$\begin{aligned} 3x+2>5 &\quad \text{or} \quad 3x+2<-5 \\ 3x>3 &\quad \text{or} \quad 3x<-7 \\ x>1 &\quad \text{or} \quad x<-\tfrac{7}{3} \end{aligned}$$

$$= (1,\infty) \cup \left(-\infty, -\frac{7}{3}\right).$$

Example 8
Solve for x

$$\begin{aligned} & 3x+4 \le 2-5x \le 7x-6 \\ & 3x+4 \le 2-5x \ \text{ and } \ -2-5x \le 7x-6 \\ \Rightarrow \ & 8x \le -2 \ \text{ and } \ -12x \le -4 \\ \Rightarrow \ & x \le -\tfrac{1}{4} \ \text{ and } \ x \ge \tfrac{1}{3} \\ & = \left(-\infty, -\tfrac{1}{4}\right] \cap \left[\tfrac{1}{3}, \infty\right) \end{aligned}$$

$$\begin{aligned} &= \phi \text{ (empty set, since they do not intersect)} \\ &= \text{No solution} \end{aligned}$$

Practice Exercise 2A

1. Simplify

   (a) $-2 < x < 5$ or $1 < x < 7$

   (b) $-1 \le x \le 4$ and $-3 < x < 2$

(c) $x < 4$ and $x > 2$

(d) $5 - 2x < 0$ and $-x < 3$

2. Solve for x:

(a) $-4 \leq 2x + 3 < 5$

(b) $4 < \dfrac{1}{3}(1 - 2x) \leq 7$

3. Solve for x:

(a) $|3x - 4| = 2$

(b) $|2x - 1| < 7$

(c) $|2x - 5| > 3$.

4. Solve for x:

$$10x - 7 < 5x + 4 \leq 6x - 9$$

Quadratic Inequalities

The graph of the quadratic function

$$y = ax^2 + bx + c, \quad a \neq 0$$

is a parabola. When $a > 0$, the parabola faces upwards. When $a < 0$, the parabola faces downwards.

Graph of $y = ax^2 + bx + c$

Example 1

Solve: $(5x + 2)(x + 4) < 0$.

Method I
Graphical Method

Graph of $y = (5x + 2)(x + 4)$

From the graph, solution is $-4 < x < \dfrac{-2}{5}$

Method II
Method of Sign Table

The critical values are where $(5x + 2)(x + 4) = 0$, i.e. where $x = \dfrac{-2}{5}$ or -4. The critical values divide the real number line into 3 segments, which we use to draw a sign table as follows:

Sign Table

	$x < -4$	$-4 < x < -\frac{2}{5}$	$x > -\frac{2}{5}$
$5x + 2$	$-$	$-$	$+$
$x + 4$	$-$	$+$	$+$
$(5x + 2)(x + 4)$	$+$	$-$	$+$

The sign in each of the factors $(5x + 2)$ and $(x + 4)$ in each segment are the same. So use a convenient point in each segment to obtain the signs.

Complete the last row of the sign table by multiplying the signs above.
From the last row of the sign table, the solution is the middle column, i.e.

$$-4 < x < -\frac{2}{5}$$

Example 2: Solve for x

$$(3+x)(4-x) \leq 0.$$

Method I: (Graphical Method)
Graph of $y = (3+x)(4-x)$

From the graph, solution is

$$x \leq -3 \;\text{ or }\; x \geq 4$$

i.e. $(-\infty, -3] \cup [4.\infty)$.

Method II (Method of Sign Table)
The critical values are where

$$(3+x)(4-x) = 0, \text{ i.e. where } x = -3 \text{ or } 4$$

These critical values divide the real number line into 3 segments, which we use to draw a sign table as follows:

Sign Table

	$x < -3$	$-3 < x < 4$	$x > 4$
$3 + x$	$-$	$+$	$+$
$4 - x$	$+$	$+$	$-$
$(3 + x)(4 - x)$	$-$	$+$	$-$

From the last row of the sign table, the solution is in the first and last columns, i.e. $x \leq -3$ or $x \geq 4$.

Practice Exercise 2B
Solve for x:

1. $(2x - 3)(3x + 1) \leq 0$
2. $(x - 5)(4 - x) \geq 0$
3. $(3 - 2x)(5 - x) > 0$
4. $(2 - x)(x + \frac{1}{2}) < 0$

Cubic Inequalities
The graph of a cubic function

$$y = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

consists of two shapes as follows:
Graph of $y = ax^3 + bx^2 + cx + d$

Example 1. Solve for x:

$$(x + 1)(x - 1)(x - 2) < 0.$$

Method I
Graphical Method

Graph of $y = (x+1)(x-1)(x-2)$

From the graph, solution is

$$x < -1 \text{ or } 1 < x < 2$$

$$(\text{i.e., } (-\infty, -1) \cup (1, 2))$$

Method II
Method of Sign Table
The critical values are where

$$(x+1)(x-1)(x-2) = 0$$

i.e., where $x = -1$ or 1 or 2.
These critical values divide the real number line into 4 segments, which we use to draw the sign table as follows:

Sign Table

	$x<-1$	$-1<x<1$	$1<x<2$	$x>2$
$x+1$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$+$	$+$
$x-2$	$-$	$-$	$-$	$+$
$(x+1)(x-1)(x-2)$	$-$	$+$	$-$	$+$

From the last row of the sign table, the solution is $x < -1$ or$1 < x < 2$.

Example 2. Solve for x:

$$(4-x)(x+3)(2x-5) \geq 0$$

Method I (Graphical Method)
Graph of $y = (4-x)(x+3)(2x-5)$

From the graph, solution is

$$x \leq -3 \text{ or } 2.5 \leq x \leq 4$$

i.e. (i.e. $(-\infty, -3] \cup [2.5, 4]$).

Method II (Method of Sign Table)
The critical values are where

$$(4-x)(x+3)(2x-5) = 0$$

i.e., where $x = 4$, -3, or 2.5.
These critical values divide the real number line into 4 segments, which we use to draw the sign table as follows:

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Sign Table

	$x < -3$	$-3 < x < 2.5$	$2.5 < x < 4$	$x > 4$
$4 - x$	$+$	$+$	$+$	$-$
$x + 3$	$-$	$+$	$+$	$+$
$2x - 5$	$-$	$-$	$+$	$+$
$(4 - x)(x + 3)(2x - 5)$	$+$	$-$	$+$	$-$

From the last row of the sign tale, the solution is $x \leq -3$ or $2.5 \leq x \leq 4$.

Practice Exercise 2C

1. $(x + 4)(x + 2)(4x + 1) > 0$
2. $(4 - x)(x - 3)(2x - 7) \geq 0$
3. $(x - 2)(x - 4)(4x - 7) \leq 0$
4. $(x + 2)(x - 3)(1 - x) < 0$

Order in the real number system
Example 1.
Prove that for all real numbers a and b,

$$a^4 + b^4 \geq 2a^2b^2$$

Consider $a^4 + b^4 - 2a^2b^2$.
Factorize: $= (a^2 - b^2)^2 \geq 0$, for all a and b.

$$\Rightarrow \quad a^4 + b^4 - 2a^2b^2 \geq 0$$

$$\Rightarrow \quad a^4 + b^4 \geq 2a^2b^2$$

Example 2

Prove that for all real numbers a, b and c,

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + a^2c^2.$$

From Example 1 above,

$$\begin{array}{ll} a^4 + b^4 \geq 2a^2b^2 & (1) \\ b^4 + c^4 \geq 2b^2c^2 & (2) \\ c^4 + a^4 \geq 2a^2c^2 & (3) \\ \hline \end{array}$$

Add $2(a^4 + b^4 + c^4) \geq 2(a^2b^2 + b^2c^2 + a^2c^2)$

$\Rightarrow \quad a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + a^2c^2$

Example 3

Prove that for all positive numbers x and y:

$$(x^7 + y^7)(x^2 + y^2) \geq (x^5 + y^5)(x^4 + y^4)$$

Consider $(x^7 + y^7)(x^2 + y^2) - (x^5 + y^5)(x^4 + y^4)$.

$$\begin{aligned} \text{Factorize:} \quad &= x^9 + x^7y^2 + x^2y^7 + y^9 \\ &\quad -(x^9 + x^5y^4 + x^4y^5 + y^9) \\ &= (x^7y^2 - x^5y^4) + (x^2y^7 - x^4y^5) \\ &= x^5y^2(x^2 - y^2) + x^2y^5(y^2 - x^2) \\ &= (x^2 - y^2)(x^5y^2 - x^2y^5) \\ &= x^2y^2(x+y)(x-y)(x^3 - y^3) \\ &= x^2y^2(x+y)(x-y)(x-y)(x^2 + xy + y^2) \\ &= x^2y^2(x+y)(x-y)^2(x^2 + xy + y^2) \\ &\geq 0 \end{aligned}$$

(since all the factors are ≥ 0)

$$\Rightarrow \quad (x^7 + y^7)(x^2 + y^2) - (x^5 + y^5)(x^4 + y^4) \geq 0$$

$$\Rightarrow \qquad (x^7 + y^7)(x^2 + y^2) \geq (x^5 + y^5)(x^5 + y^4).$$

Practice Exercise 2D

Prove that for all real numbers a, b, c and d:

1. $a^2 + b^2 \geq 2ab$
2. $(a + b)^2 \geq 4ab$
3. $(ab + cd)^2 \leq (a^2 + c^2)(b^2 + d^2)$
4. $a^4 + b^4 + c^4 + d^4 \geq 4abcd$.
   [Hint: Use $a^4 + b^4 \geq 2a^2b^2$ and $a^2b^2 + c^2d^2 \geq 2abcd$].
5. Prove that if a and b are positive real numbers, then

$$2(a^9 + b^9) \geq (a^4 + b^4)(a^5 + b^5).$$

Summary
We use the interval notation and segments of the real number line to represent inequalities, their union and intersection; as well as those involving absolute values of numbers.
We use properties of inequalities to solve linear inequalities and problems of order in the real number system. We then adopt the graphical method and the method of the sign table to solve quadratic and cubic inequalities.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 38-68.

LECTURE THREE

Rational Functions and Curve Sketching

Introduction

We shall study how to perform the basic operations of addition, subtraction, multiplication and division on rational functions.
We shall then study how to split or resolve or decompose certain rational functions into a sum of partial fractions.

We shall conclude by studying how to find the domain and range and how to draw the graphs of linear, quadratic and some cubic and rational functions.

Objectives

The reader should be able to

- perform the basic algebraic operations on rational functions;
- split or resolve or decompose certain rational functions into a sum of partial fractions; and
- find the domain and range, and draw the graphs, of linear, quadratic and some cubic and rational functions.

Pre-Test

1. Reduce to its lowest terms

   (a) $\dfrac{ax - ay}{y^2 - x^2}$ (b) $\dfrac{xy - x^2}{x - y}$

2. Simplify

   (a) $\dfrac{1}{b} - \dfrac{2}{ab^2} + \dfrac{3}{a^2b^3}$

   (b) $\dfrac{x}{x+y} - \dfrac{y}{x-y} + \dfrac{2y^2}{y^2 - x^2}$

   (c) $\left(\dfrac{a^2}{b} + \dfrac{b^2}{a}\right)\left(\dfrac{ab}{a+b}\right)$

   (d) $\left(\dfrac{1}{x} - \dfrac{1}{y}\right)\Big/ (x-y)^2$

3. Resolve into a sum of partial fractions

   (a) $\dfrac{x}{(x+5)(x+3)}$ (b) $\dfrac{x^2+1}{x^2-1}$ (c) $\dfrac{x^2}{(x-2)^2}$

4. Find the maximal domain and the corresponding range of:

   (a) $y = 3x - 2$ (b) $y = 3$
   (c) $x = 5$ (d) $y = x^2 - 8x + 15$
   (e) $y = -x^2 + 3x - 2$ (f) $y = (x-1)(x+1)(x+4)$
   (g) $y = \dfrac{2x+1}{3-4x}$ (h) $y = \dfrac{x^2}{1-x^2}$

5. Sketch the graphs of each of the functions in Question 3 above.

Addition and Subtraction of Rational Functions

A function of the form $\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials, is called a rational function:

e.g., $\dfrac{2}{x}$, $\dfrac{x-4}{x^2+2x-3}$, $\dfrac{x^3}{x^2-1}$.

Example 1.
Reduce to its lowest terms:
$$\frac{1-x^2}{x-x^2}.$$
Factorize the numerator and denominator:
$$\frac{1-x^2}{x-x^2} = \frac{(1+x)(1-x)}{x(1-x)} = \frac{1+x}{x}.$$

Example 2
Simplify: $\dfrac{2a}{a+2} - \dfrac{a}{a-3}$
L.C.M $= (a+2)(a-3)$
$$\frac{2a}{a+2} - \frac{a}{a-3} = \frac{2a(a-3) - a(a+2)}{(a+2)(a-3)}$$
$$= \frac{2a^2 - 6a - a^2 - 2a}{(a+2)(a-3)} = \frac{a(a-8)}{(a+2)(a-3)}.$$

Example 3
Simplify: $\dfrac{3}{x^2-x} - \dfrac{7}{x^2-1} + \dfrac{3}{x^2+x}$.
Factorize the denominators and take the L.C.M.
$$\frac{3}{x^2-x} - \frac{7}{x^2-1} + \frac{3}{x^2+x} = \frac{3}{x(x-1)} - \frac{7}{(x+1)(x-1)} + \frac{3}{x(x+1)}$$
$$= \frac{3(x+1) - 7x + 3(x-1)}{x(x-1)(x+1)} = \frac{-x}{x(x-1)(x+1)}$$
$$= \frac{-1}{(x-1)(x+1)}$$

Multiplication and Division of Rational Functions
Example 4
Simplify:
$$\left(\frac{x+2}{1-x^2}\right)\left(\frac{x-1}{x^2+5x+6}\right)$$

Factorize:

$$\frac{x+2}{1-x^2} \times \frac{x-1}{x^2+5x+6} = \frac{(x+2)}{(1+x)(1-x)} \times \frac{(x-1)}{(x+2)(x+3)}$$
$$= \frac{-1}{(1+x)(x+3)}$$

Example 5
Simplify

$$\left(\frac{a^3+b^3}{a-b}\right) \Big/ \left(\frac{a+b}{a-b}\right) = \frac{a^3+b^3}{a-b} \times \frac{a-b}{a+b}$$
$$= \frac{(a+b)(a^2-ab+b^2)(a-b)}{(a-b)(a+b)} = a^2-ab+b^2$$

Practice Exercise 3A

1. Reduce to its lowest terms:
   (a) $\dfrac{2x^2-2y^2}{x^2+2xy+y^2}$ (b) $\dfrac{3ac-bc}{9a^2-b^2}$
   (c) $\dfrac{a^2-4x^2}{(2x-a)^2}$ (d) $\dfrac{a^2-1}{a^6-1}$

2. Simplify: (a) $\dfrac{a}{b}+\dfrac{b}{c}+\dfrac{c}{a}$ (b) $1-\dfrac{a}{a+b}$
   (c) $\dfrac{a}{a+b}+\dfrac{b}{b-a}$ (d) $\dfrac{3}{x+3}+\dfrac{5}{x^2-9}$
   (e) $\dfrac{x+y}{2(x-y)}+\dfrac{x-y}{2(x+y)}-\dfrac{x^2+y^2}{x^2-y^2}$.

3. Simplify:
   (a) $\left(\dfrac{a-b}{a+b}\right)\left(\dfrac{a^2}{b^2}-\dfrac{b^2}{a^2}\right)$
   (b) $\left(\dfrac{x}{x+y}+\dfrac{y}{x-y}\right)\left(\dfrac{y^2-x^2}{2y^2}\right)$
   (c) $\left(\dfrac{1}{x}-\dfrac{1}{y}\right)\Big/\left(\dfrac{1}{x}+\dfrac{1}{y}\right)$

(d) $\left(\dfrac{x-y}{x}\right) \Big/ \left(\dfrac{x^2-y^2}{xy}\right)$

Partial Fractions

If (degree of $f(x)$)<(degree of $g(x)$), then the rational function $\dfrac{f(x)}{g(x)}$ is called a PROPER rational function.

Properties

1. By the Division Algorithm, any rational function can be expressed as a sum of a polynomial (possibly the zero polynomial) and a proper rational function,
   i.e., $\dfrac{f(x)}{g(x)} = q(x) + \dfrac{r(x)}{g(x)}$
   where $\dfrac{r(x)}{g(x)}$ is a proper rational function..

2. If the polynomials $g(x)$ and $h(x)$ do not have any common factor of degree ≥ 1, then

$$\frac{f(x)}{g(x)h(x)} = q(x) + \frac{r(x)}{g(x)} + \frac{s(x)}{h(x)}$$

   where $\dfrac{r(x)}{g(x)}$ and $\dfrac{s(x)}{h(x)}$ are proper rational functions.

Remark: Property (2) above is called the **splitting** or **resolution** or **decomposition** of a rational function into a sum of partial fractions.

3. $\dfrac{px+q}{(x-a)^2} = \dfrac{p}{x-a} + \dfrac{pa+q}{(x-a)^2}$
   i.e. Expressing a rational function with repeated linear factor in the denominator as a sum of partial fractions with constant numerators.

Example 1: Resolve into a sum of partial fractions:

$$\frac{x+5}{(x+1)(x+4)}$$

$$\frac{x+5}{(x+1)(x+4)} = \frac{A}{x+1} + \frac{B}{x+4} \tag{1}$$

where A and B are constants to be determined.
Note:

1. The denominators of $\dfrac{A}{x+1}$ and $\dfrac{B}{x+4}$ are of degree 1. Since $\dfrac{A}{x+1}$ and $\dfrac{B}{x+4}$ are proper rational functions A and B must be of degree 0, i.e. constants.

2. Also the given rational function is a proper rational function. So we do not need to use the division algorithm.
   Equation (1) becomes:

$$\begin{aligned} \frac{x+5}{(x+1)(x+4)} &= \frac{A(x+4)+B(x+1)}{(x+1)(x+4)} \\ \Rightarrow \quad x+5 &\equiv A(x+4)+B(x+1) \qquad (2) \end{aligned}$$

Convenient values of x to substitute are where $x+4=0$ and $x+1=0$, i.e. $x=-4$ and$x=-1$.
Put $x=-4$ in (2):

$$\begin{aligned} -4+5 &= 0+B(-4+1) \\ 1 &= -3B, \quad B = -\frac{1}{3} \end{aligned}$$

Put $x=-1$ in (2):

$$\begin{aligned} -1+5 &= A(-1+4)+0 \\ 4 &= 3A, \quad A = \frac{4}{3} \end{aligned}$$

Therefore,

$$\frac{x+5}{(x+1)(x+4)} = \frac{4/3}{x+1} - \frac{1/3}{x+4}$$

Remark.
You may also substitute any 2 different values of x in equation (2) above and solve the resultant simultaneous equations in A and B.

Example 2
Split into a sum of partial fractions:

$$\frac{x^3+x+1}{x^2-1}.$$

First use the division algorithm to reduce the given improper fraction into a sum of a polynomial and a proper rational function

$$\begin{array}{rl} & \quad x \\ \underline{x^2-1} & \overline{|x^3+x+1} \\ & \;-(x^3-x) \\ \hline & \qquad\quad 2x+1 \end{array}$$

$$\Rightarrow \quad \frac{x^3+x+1}{x^2-1} = x + \frac{2x+1}{x^2-1}.$$

Now,

$$\begin{aligned} \frac{2x+1}{x^2-1} &= \frac{2x+1}{(x+1)(x-1)} \\ \frac{2x+1}{(x+1)(x-1)} &= \frac{A}{x+1} + \frac{B}{x-1} \\ \Rightarrow \quad \frac{2x+1}{(x+1)(x-1)} &= \frac{A(x-1)+B(x+1)}{(x+1)(x-1)} \\ \Rightarrow \quad 2x+1 &\equiv A(x-1)+B(x+1) \end{aligned}$$

Convenient values of x to substitute are $x = 1$ and$x = -1$.
Put $x = 1$:

$$\begin{aligned} 2+1 &= 0 + B(1+1) \\ 3 &= 2B, \quad B = \frac{3}{2} \end{aligned}$$

Put $x = -1$:

$$\begin{aligned} -2+1 &= A(-1-1)+0 \\ -1 &= -2A,\ A = \frac{1}{2} \\ \Rightarrow \quad \frac{x^3+x+1}{x^2-1} &= x + \frac{1/2}{x+1} + \frac{3/2}{x-1} \end{aligned}$$

Example 3

Express $\dfrac{x^2-x+2}{(x-1)^2}$ as a sum of partial fractions with constant numerators.

First use division algorithm to reduce the improper rational function to a proper one.

$$\begin{array}{r|l} & \underline{1\qquad\qquad} \\ x^2-2x+1 & x^2-x+2 \\ & \underline{-(x^2-2x+1)} \\ & \qquad\quad x+1 \end{array}$$

$$\begin{aligned} \frac{x^2-x+2}{(x-1)^2} &= 1 + \frac{x+1}{(x-1)^2},\ p=1, q=1, a=1 \\ &= 1 + \frac{p}{x-1} + \frac{pa+q}{(x-1)^2} \\ &= 1 + \frac{1}{x-1} + \frac{2}{(x-1)^2} \end{aligned}$$

Practice Exercise 3B

Resolve into partial fractions:

(1) $\dfrac{3x+1}{(2+x)(x-3)}$ (2) $\dfrac{2x^2}{(x-3)(x-2)}$

(3) $\dfrac{x^3}{(x-2)^2}$ (4) $\dfrac{x}{x^2-4}$

(5) $\dfrac{x^3+x+1}{x^2-1}$ (6) $\dfrac{3x-5}{(x-4)^2}$

(7) $\dfrac{2x^2}{2x^2-3x-2}$ (8) $\dfrac{4x^2+x+6}{4x^2-4x-3}$

Domain and Range

Determination of the Domain

The **maximal domain** or the **largest** (**possible**) **domain** of a function $y = f(x)$ is the set of all real values for which the function is defined.
Note that the domain of a function can be restricted to a non-empty subset of the maximal domain.
The domain of a function $y = f(x)$ answers the question; "when is the function defined?"

Example 1

Linear Function	Maximal Domain
$y = ax + b,\ a \neq 0$	$\mathbb{R}$
$y = b$	$\mathbb{R}$
$x = a$	$\{a\}$

Example 2

The maximal domain of all polynomial functions,$y = f(x)$ is $\mathbb{R}$, the set of all real numbers. If

$$y = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

then for any $x \in \mathbb{R}$, y is well-defined.

Example 3

The maximal domain of a rational function $y = \dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions, is

$$\mathbb{R}\backslash\{x \in \mathbb{R} | g(x) = 0\}$$

e.g. Maximal domain of $y = \dfrac{2x}{(x-1)(x+2)}$ is $\mathbb{R}\backslash\{1, -2\}$.

Determination of the Range

The range of a function $y = f(x)$ is the set

$$\text{Range } (f) = \{f(x) | x \in \text{ Domain } (f)\}.$$

Hence the range or the image of a function is the set of all images of the function.
One way of finding the range of $y = f(x)$ is to express x as a function of y, $x = g(y)$, and then the range is the set of all the real values of y for which the function g is defined.

Example 4
Find the range of the linear function.

$$y = ax + b, \quad a \neq 0$$

which corresponds to the maximal domain.
Make x the subject of the formula

$$\begin{aligned} & y = ax + b \Rightarrow ax = y - b \\ \Rightarrow \quad & x = \frac{y-b}{a} \end{aligned}$$

x is defined for all real values of y
$\Rightarrow$ Range $= \mathbb{R}$.

Example 5
Find the range of the quadratic function: $y = ax^2 + bx + c$, $a \neq 0$.
Try to solve for x:

$$ax^2 + bx + (c - y) = 0$$

x is well-defined if the discriminant of the last equation is ≥ 0, i.e.

$$b^2 - 4a(c - y) \geq 0$$

$$\Rightarrow \quad 4ay \geq 4ac - b^2 \tag{1}$$

If $a > 0$, (1) becomes $y \geq \dfrac{4ac - b^2}{4a}$.
Hence the range of $y = ax^2 + bx + c$, when $a > 0$ is $\left[\dfrac{4ac - b^2}{4a}, \infty\right)$.
If $a < 0$, (1) becomes

$$y \leq \frac{4ac - b^2}{4a}$$

Hence the range of $y = ax^2 + bx + c$, when $a < 0$ is $\left(-\infty, \dfrac{4ac - b^2}{4a}\right]$.

Remarks

1. Domain is the part of the x-axis occupied by the function.
2. Range is the part of the y-axis occupied by the function.
3. The range of a cubic function is $\mathbb{R}$.

Example 6
Find the range of the rational function

$$y = \frac{3x + 2}{4x - 5}$$

Make x the subject of the formula

$$\begin{aligned} y(4x - 5) &= 3x + 2 \\ (4y - 3)x &= 2 + 5y \\ x &= \frac{2 + 5y}{4y - 3} \end{aligned}$$

x is defined for all y except $y = \dfrac{3}{4}$.
$\Rightarrow$ Range $= \mathbb{R} \backslash \left\{\dfrac{3}{4}\right\}$.

Example 7
Find the range of the rational function

$$y = \frac{x + 1}{x^2 + 2x + 2}.$$

Try to solve for x:

$$y(x^2 + 2x + 2) = x + 1$$

$$yx^2 + (2y - 1)x + (2y - 1) = 0$$

x is well defined if the discriminant of the quadratic equation in x is ≥ 0

$$\begin{aligned}(2y-1)^2 - 4y(2y-1) &\geq 0 \\ 4y^2 - 4y + 1 - 8y^2 + 4y &\geq 0 \\ 4y^2 - 1 \leq 0,\ (2y++1)(2y-1) &\leq 0 \\ -\frac{1}{2} \leq y &\leq \frac{1}{2}\end{aligned}$$

$\Rightarrow$ Range $= \left[-\frac{1}{2}, \frac{1}{2}\right]$.

Example 8
Find the range of the rational function

$$y = \frac{x^2 + 3x + 6}{x + 1}$$

Try to solve for x.

$$\begin{aligned}&y(x+1) = x^2 + 3x + 6 \\ &x^2 + (3-y)x + (6-y) = 0\end{aligned}$$

x is defined if $(3-y)^2 - 4(6-y) \geq 0$

$$\begin{aligned}&9 - 6y + y^2 - 24 + 4y \geq 0 \\ &y^2 - 2y - 15 \geq 0 \\ &(y-5)(y+3) \geq 0\end{aligned}$$

$\Rightarrow \quad y \leq -3$ or $y \geq 5$
$\Rightarrow$ Range $= (-\infty, -3] \cup [5, \infty)$.

Example 9
Find the range of the rational function

$$y = \frac{x^2 - x + 1}{x^2 + x - 1}$$

Try to solve for x:

$$\begin{aligned}&y(x^2 + x - 1) = x^2 - x + 1 \\ &(y-1)x^2 + (y+1)x - (y+1) = 0\end{aligned}$$

x is defined if

$$\begin{aligned}
&(y+1)^2 + 4(y-1)(y+1) \geq 0 \\
&(y+1)(y+1+4y-4) \geq 0 \\
&(y+1)(5y-3) \geq 0
\end{aligned}$$

$\Rightarrow \quad y \leq -1$ or $y \geq \dfrac{3}{5}$

$\Rightarrow$ Range $= (-\infty, -1] \cup \left[\dfrac{3}{5}, \infty\right)$.

Practice Exercise 3C

Find the maximal domain and the corresponding range:

(1)	$y = 2 - 5x$	(2)	$y = 9$
(3)	$x = 4$	(4)	$y = x^2 - 3x + 2$
(5)	$y = 4 + 3x - x^2$	(6)	$y = x^2 - 4$
(7)	$y = \dfrac{3-4x}{2x+1}$	(8)	$y = \dfrac{3x+5}{4x-7}$
(9)	$y = \dfrac{4x-7}{x^2-6x+8}$	(10)	$y = \dfrac{x^2+2}{2x+1}$
(11)	$y = \dfrac{4x^2+4x+1}{x^2-4}$	(12)	$y = \dfrac{x^2-5x+4}{x-5}$
(13)	$y = \dfrac{x^2-2x+4}{x^2+2x+4}$	(14)	$y = \dfrac{x^2+1}{x^2+x+1}$

Graphs of Linear Functions

Two points are sufficient to sketch the graph of a linear function.

Example 1

Sketch the graph of:

(a) $y = 4$ (b) $x = -2$

Solution

(a) $y = 4$ is a line parallel to the x-axis.

(b) $x = -2$ is a line parallel to the y-axis

(a) (b)

Graph of $y = 4$ **Graph of** $x = -2$

Example 2
Sketch the graph of:
(a) $y = 3x - 2$ (b) $y = 5 - 2x$

(a) $y = 3x - 2$
When $x = 0$, $y = -2$
When $y = 0$, $x = \dfrac{2}{3}$
$\Rightarrow$ $(0, -2)$ and $\left(\dfrac{2}{3}, 0\right)$ are two points on the graph. Plot the points and join them with a straight line

(b) $y = 5 - 2x$
When $x = 0$, $y = 5$
When $y = 0$, $x = 2.5$
$\Rightarrow$ (0.5) and $(2.5, 0)$ are two points on the graph.

.

Example 3
Sketch the graph of

$$y = (x+2)(x-3).$$

Intercepts: The points where the graph intersects the x-axis are called the x-intercepts. They are found by solving the equation $y = 0$, i.e.
$(x+2)(x-3 = 0 \Rightarrow x = -2$ or 3.
Hence the points $(-2, 0)$ and$(3, 0)$ are the x-intercepts.
The points where the graph intersects the y-axis are called the y-intercepts. They are found by solving the equation $x = 0$, i.e.

$$y = (0+2)(0-3) = -6.$$

Hence the point $(0, -6)$ is the only y-intercept.

Vertex and Line of symmetry (or axis):
Complete the square:

$$\begin{aligned} y &= (x+2)(x-3) = x^2 - x - 6 \\ &= \left(x - \frac{1}{2}\right)^2 - 6 - \frac{1}{4} = \left(x - \frac{1}{2}\right)^2 - \frac{25}{4}. \end{aligned}$$

Hence the vertex is at the point

$$(x, y) = \left(\frac{1}{2}, -\frac{25}{4}\right).$$

The line of symmetry is the vertical line $x = \frac{1}{2}$.
Note that the graph is a parabola facing upwards.

Graph of $y = (x+2)(x-3)$

Join the points in a curve by free-hand, not by straight lines.

Example 4
Sketch the graph of

$$y = (4-x)(x+3)$$

x**-intercepts**: Put $y = 0$, i.e.

$$(4-x)(x+3) = 0 \Rightarrow x = 4 \text{ or } -3$$

Hence, $(4,0)$ and $(-3,0)$ are the x-intercepts.
y**-intercepts**: Put $x = 0$, i.e.

$$y = (4-0)(0+3) = 12.$$

Hence $(0,12)$ is the only y-intercept.

Vertex and axis
Complete the square:

$$\begin{aligned} y &= (4-x)(x+3) = 12 + x - x^2 \\ &= 12 - (x^2 - x) \\ &= 12 + \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \end{aligned}$$

$$= \frac{49}{4} - \left(x - \frac{1}{2}\right)^2$$

Hence the vertex is at the point

$$\left(\frac{1}{2}, \frac{49}{4}\right).$$

The axis is the line $x = \dfrac{1}{2}$.

Note that the graph is a parabola facing downwards.

Graph of $y = (4 - x)(x + 3)$

Example 5

Sketch the graph of

$$y = (x + 2)(x - 1)(x - 4).$$

x**-intercepts**: Put $y = 0$, i.e.

$$(x + 2)(x - 1)(x - 4) = 0 \Rightarrow x = -2, 1 \text{ or } 4.$$

Hence $(-2, 0)$, $(1, 0)$ and $(4, 0)$ are the x-intercepts.

y**-intercepts**: Put $x = 0$, i.e.

$$y = (0 + 2)(0 - 1)(0 - 4) = 8.$$

Hence $(0, 8)$ is the only y-intercept.

Sketch the graph using the general shape of a cubic function

$$y = ax^3 + bx^2 + cx + d, \quad \text{for} \quad a > 0.$$

Graph of $y = (x+2)(x-1)(x-4)$

Note: It is possible to use calculus to determine the coordinates of the vertices.

Example 6
Sketch the graph of
$$y = (x+4)(x+1)(2-x).$$
x**-intercepts**: Put $y = 0$, i.e.
$$(x+4)(x+1)(2-x) = 0 \Rightarrow x = -4, -1 \text{ or } 2$$
Hence $(-4,0), (-1,0)$ and$(2,0)$ are the x-intercepts.
y**-intercepts**: Put $x = 0$, i.e.
$$y = (0+4)(0+1)(2-0) = 8$$
Hence $(0,8)$ is the only y-intercept.
Sketch the graph using the general shape of a cubic function
$$y = ax^3 + bx^2 + cx + d, \text{ for } a < 0.$$

Graph of $y = (x+4)(x+1)(2-x)$

Graphs of Rational Functions

Apart from the intercepts, graphs of rational functions have a special feature, called asymptotes.

An asymptote is a function to which the graph of a given function approaches. We shall be considering asymptotes which are straight lines parallel to the x-axis and the y-axis, called Horizontal and Vertical asymptotes. Hence the graph of the rational function approaches these lines as x approaches $+\infty$ or $-\infty$ or as y approaches $+\infty$ or $-\infty$.

Determination of the asymptotes

Let $y = \dfrac{f(x)}{g(x)}$. Then the vertical asymptotes are obtained from values of x for which $g(x) = 0$, where the function is NOT defined.
Using the Division Algorithm, we can express the rational function in the form

$$y = \frac{f(x)}{g(x)} = q(x) + \frac{r(x)}{g(x)}$$

where $g(x)$ is a polynomial and $\dfrac{r(x)}{g(x)}$ is a proper rational function. Then $y = q(x)$ is an asymptote. If $q(x)$ is a constant, then the asymptote is horizontal.
Use dotted lines to represent the asymptotes.

Example 7

Sketch the graph of $y = \dfrac{3}{2-x}$.

Intercepts

$x = 0 \;\Rightarrow\; y = \dfrac{3}{2-0} = \dfrac{3}{2}$

$\Rightarrow \left(0, \dfrac{3}{2}\right)$ is a y-intercept.

$y = 0 \Rightarrow$ no value of x.

$\Rightarrow$ there is no x-intercept.

Vertical asymptotes: The line $x = 2$.

Horizontal asymptote: Since $y = 0 + \dfrac{3}{2-x}$, it follows that $y = 0$ is a horizontal asymptote.

Graph of $y = \dfrac{3}{2-x}$

Explanation:

1. Since the graph must approach the asymptotes, and must not cross the vertical asymptote, and must pass through the y-intercept $\left(0, \dfrac{3}{2}\right)$, then we obtain the branch of the graph to the left of the vertical asymptote.

2. Pick a convenient x-coordinate to the right of the vertical asymptote to determine in which region the other branch of the graph will lie. For example, put $x = 3$, then $y = \dfrac{3}{2-3} = -3$.

Example 8

Sketch the graph of $y = \dfrac{2x+1}{x-3}$.

Intercepts

$$x = 0 \Rightarrow y = \frac{0+1}{0-3} = -\frac{1}{3}$$

$\Rightarrow \left(0, -\dfrac{1}{3}\right)$ is a y-intercept

$$y = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = -\frac{1}{2}$$

$\Rightarrow \left(\dfrac{1}{2}, 0\right)$ is an x-intercept.

Vertical asymptote: The line $x = 3$.

Horizontal asymptote: Use division algorithm:

$$\begin{array}{r|l} & \quad 2 \\ \cline{2-2} \underline{x-3} & 2x+1 \\ & \underline{-(2x-6)} \\ & \quad 7 \end{array}$$

$\Rightarrow \ y = 2 + \dfrac{7}{x-3}$

$\Rightarrow \ y = 2$ is a horizontal asymptote.

Graph of $y = \dfrac{2x+1}{x-3}$

Example 9

Sketch the graph of $y = \dfrac{6}{(2-x)(x+3)}$.

Intercepts:

$$x = 0 \Rightarrow y = \frac{6}{(2-0)(0+3)} = 1$$

$\Rightarrow (0, 1)$ is a y-intercept.
$y = 0 \Rightarrow$ no value of x
$\Rightarrow$ There is no x-intercept.

Vertical asymptotes.
The lines $x = 2$ and $x = -3$.
Horizontal asymptotes: $y = 0$

Graph of $y = \dfrac{6}{(2-x)(x+3)}$

Example 10
Sketch the graph of $y = \dfrac{8(x-3)}{(x-2)(x-4)}$.
Intercepts

$$x = 0 \Rightarrow y = \frac{8(0-3)}{(0-2)(0-4)} = -3$$

$\Rightarrow (0, -3)$ is a y-intercept.
$y = 0 \Rightarrow 8(x-3) = 0 \Rightarrow x = 3$
$\Rightarrow (3, 0)$ is an x-intercept.
Vertical asymptotes: The lines are $x = 2$ and $x = 4$.
Horizontal asymptote: The line is $y = 0$.

Graph of $y = \dfrac{8(x-3)}{(x-2)(x-4)}$

Example 11

Sketch the graph of $y = \dfrac{x^2+x-6}{(x+2)(x-3)}$.

Intercepts

$$x = 0 \Rightarrow y = \frac{-6}{(-6} = 1$$

$\Rightarrow (0,1)$ is a y-intercept.
$y = 0 \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x-2)(x+3) = 0$
$\Rightarrow x = 2$ or -3
$\Rightarrow (2,0)$ and $(-3,0)$ are x-intercepts.

Vertical asymptotes: The lines are $x = -2$ and $x = 3$.

Horizontal asymptote: Use division algorithm.

$$(x+2)(x-3) = x^2 - x - 6.$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \underline{x^2 - x - 6} & x^2 + x - 6 \\ & \underline{-(x^2 - x - 6)} \\ & \quad 2x \end{array}$$

$\Rightarrow y = 1 + \dfrac{2x}{(x+2)(x-3)}$
$\Rightarrow y = 1$ is a horizontal asymptote.

Graph of $y = \dfrac{x^2 + x - 6}{(x+2)(x-3)}$

Practice Exercise 3D

Sketch the graph of each of the following functions:

(1) $y = 3 - 9x$

(2) $y = -5$

(3) $x = -4$

(4) $y = (x-2)(x-1)$

(5) $y = (4-x)(1+x)$

(6) $y = \dfrac{3}{2x+1}$

(7) $y = (1-x)(2+x)(4-x)$

(8) $y = (2-x)(x+1)(x-4)$

(9) $y = \dfrac{3x+5}{4x-7}$

(10) $y = \dfrac{8}{(x-4)(x-2)}$

(11) $y = \dfrac{x}{(x-1)(x+4)}$

(12) $y = \dfrac{x^2+1}{(x+1)(x-3)}$

(13) $y = \dfrac{1}{x}$

(14) $y = \dfrac{1}{x^2}$

(15) $y = \dfrac{1}{x^2-1}$

Summary

We are able to

- add, subtract, multiply and divide rational functions
- split certain rational functions into a sum of partial fractions
- find the domain and range, as well as sketch the graphs of linear, quadratic and some cubic and rational functions.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 69-119.

LECTURE FOUR

Mathematical Induction

Introduction

When a mathematical statement $P(n)$ is true for all natural numbers $n = 1, 2, 3, \ldots$, it is not possible to verify the statement using a case by case method, because we have an infinite number of cases to verify and we can never finish.

The principle of mathematical induction is a simple but powerful tool for proving such mathematical statements.

Objective

The reader should be able to apply the principle of mathematical induction to prove several types of mathematical statements, which are true for all natural numbers.

Pre-Test

Prove, by mathematical induction; that for all natural numbers $n = 1, 2, 3, \ldots$

1. $1(3) + 2(4) + 3(5) + \cdots + n(n+2) = \frac{1}{6}n(n+1)(2n+7)$
2. $\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

3. $n(n+1)(n+2)$ is divisible by 6

4. 3 is a factor of $n(n^2+2)$

5. 9 divides $2^{2n} - 3n - 1$

6. 37 divides $10^{3n} - 1$

The Principle
The principle of mathematical induction can be stated as follows:
Let $P(n)$ be a mathematical statement defined for any natural number $n = 1, 2, 3, \ldots$
Then $P(n)$ is true for all natural numbers n if the following 3 steps are satisfied.
Step I: Show $P(1)$ is true.
Step II: Assume, as an inductive hypothesis, that $P(k)$ is true for some $n = k$.
Step III: Using Step II above, show that $P(k+1)$ is true.

Explanation
If Step I is established, i.e. if $P(1)$ is true, then Steps II and III show that $P(2)$ is true. Steps II and III again show that $P(3)$ is true. Continue this process and we have that $P(1), P(2), P(3), \cdots, P(n), \cdots$ are true for all n.

Example 1
Use mathematical induction to prove that for all natural numbers n,

$$1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Step I. When $n = 1$,
L.H.S. = 1, RHS = $1^2 = 1$ = LHS
$\Rightarrow$ The formula is true for $n = 1$.
Step II. Assume that the formula is true for $n = k$, i.e. assume

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

Step III. Consider $n = k + 1$.

$$\begin{aligned} L.H.S. &= 1 + 3 + 5 + \cdots + (2k-1) + (2k+1) \\ &= [1 + 3 + 5 + \cdots + (2k-1)] + 2k + 1 \\ &= k^2 + 2k + 1, \text{ using the assumption in Step II} \\ &= (k+1)^2 = R.H.S. \end{aligned}$$

$\Rightarrow$ The formula is true for $n = k + 1$.
Hence by the principle of mathematical induction, the formula is true for all natural numbers n.

Example 2
Use mathematical induction to prove that for all natural numbers n,

$$1(2) + 2(3) + 3(4) + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2).$$

Step I. When $n = 1$

$$\begin{aligned} L.H.S. &= 1(2) = 2 \\ R.H.S. &= \frac{1}{3}(2)(3) = \frac{6}{3} = 2 = L.H.S. \end{aligned}$$

$\Rightarrow$ The formula is true for $n = 1$

Step II. Assume that the formula is true for $n = k$, i.e. assume

$$1(2) + 2(3) + 3(4) + \cdots + k(k+1) = \frac{1}{3}k(k+1)(k+2)$$

Step III. Consider $n = k + 1$

$$\begin{aligned} L.H.S. &= 1(2) + 2(3) + 3(4) + \cdots + k(k+1) + (k+1)(k+2) \\ &= [1(2) + 2(3) + 3(4) + \cdots + k(k+1)] + (k+1)(k+2) \\ &= \frac{1}{3}k(k+1)(k+2) + (k+1)(k+2), \text{ using the assumption in Step II.} \\ &= \frac{1}{3}(k+1)(k+2)[k+3] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{3}(k+1)(k+2)(k+3) \\
R.H.S. &= \frac{1}{3}(k+1)(k+1+1)(k+2+2) \\
&= \frac{1}{3}(k+1)(k+2)(k+3) = L.H.S.
\end{aligned}
$$

$\Rightarrow$ The formula is true for $n = k+1$.
Hence by the principle of mathematical induction, the formula is true for all natural numbers n.

Example 3
Use mathematical induction to prove that for all natural numbers n,

$$\frac{1}{2(5)} + \frac{1}{5(8)} + \frac{1}{8(11)} + \cdots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{2(3n+2)}$$

Step I. When $n = 1$,

$$
\begin{aligned}
L.H.S. &= \frac{1}{2(5)} = \frac{1}{10} \\
R.H.S. &= \frac{1}{2(3+2)} = \frac{1}{10} = L.H.S.
\end{aligned}
$$

$\Rightarrow$ The formula is true for $n = 1$.

Step II. Assume that the formula is true for $n = k$, i.e. assume

$$\frac{1}{2(5)} + \frac{1}{5(8)} + \frac{1}{8(11)} + \cdots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{2(3k+2)}$$

Step III. Consider $n = k+1$

$$
\begin{aligned}
L.H.S. &= \frac{1}{2(5)} + \frac{1}{5(8)} + \frac{1}{8(11)} + \cdots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{(3k+2)(3k+5)} \\
&= \left[\frac{1}{2(5)} + \frac{1}{5(8)} + \frac{1}{8(11)} + \cdots + \frac{1}{(3k-1)(3k+2)}\right] + \frac{1}{(3k+2)(3k+5)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)}, \text{ using the assumption in Step II} \\
&= \frac{k(3k+5)+2}{2(3k+2)(3k+5)} = \frac{3k^2+5k+2}{2(3k+2)(3k+5)} \\
&= \frac{(3k+2)(k+1)}{2(3k+2)(3k+5)} = \frac{k+1}{2(3k+5)} \\
R.H.S. &= \frac{k+1}{2(3k+3+2)} = \frac{k+1}{2(3k+5)} = L.H.S.
\end{aligned}
$$

$\Rightarrow$ The formula is true for $n = k+1$.
Hence, by the principle of mathematical induction, the formula is true for all natural numbers n.

Example 4.
Use mathematical induction to prove that

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

Step I. When $n = 1$,

$$
\begin{aligned}
L.H.S. &= 1^3 = 1 \\
R.H.S. &= \frac{1^2(2^2)}{4} = 1 = L.H.S.
\end{aligned}
$$

$\Rightarrow$ The formula is true for $n = 1$

Step II. Assume that the formula is true for $n = k$, i.e. assume

$$1^3 + 2^3 + 3^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

Step III. Consider $n = k+1$

$$
\begin{aligned}
L.H.S. &= 1^3 + 2^3 + 3^3 + \cdots + k^3 + (k+1)^3 \\
&= [1^3 + 2^3 + 3^3 + \cdots + k^3] + (k+1)^3 \\
&= \frac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ using the assumption in Step II}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{4}(k+1)^2[k^2+4(k+1)] \\
&= \frac{1}{4}(k+1)^2(k+2)^2 \\
R.H.S. &= \frac{1}{4}(k+1)^2(k+1+1)^2 \\
&= \frac{1}{4}(k+1)^2(k+2)^2 = L.H.S.
\end{aligned}
$$

$\Rightarrow$ The formula is true for $n = k+1$.
Hence, by the principle of mathematical induction, the formula is true for all natural numbers n.

Example 5
Prove that for all natural numbers n, $5^{2n} - 1$ is an integer multiple of 24.
Use mathematical induction:
Step I. When $n = 1$,

$$5^{2n} - 1 = 5^2 - 1 = 24 = 24(1).$$

$\Rightarrow$ The statement is true for $n = 1$.

Step II. Assume that the statement is true for $n = k$, i.e. assume $5^{2k} - 1$ is an integer multiple of 24, so that

$$5^{2k} - 1 = 24M$$

for some natural number M.

Step III. Consider $n = k+1$. Then

$$
\begin{aligned}
5^{2n} - 1 &= 5^{2(k+1)} - 1 = 5^{2k} \cdot 5^2 - 1 \\
&= 25(24M+1) - 1, \text{ using the assumption in Step II} \\
&= 25(24M) + 24 \\
&= 24[25M+1], \text{ which is a multiple of 24}
\end{aligned}
$$

$\Rightarrow$ The statement is true for $n = k+1$.
Hence, by the principle of mathematical induction, the statement is true for

all natural numbers n.

Example 6
Prove that for all natural numbers n,

$$9 \text{ is a factor of } 5^{2n} + 3n - 1$$

Use mathematical induction.

Step I. When $n = 1$,

$$5^{2n} + 3n - 1 = 5^2 + 3 - 1 = 27 = 9 \times 3$$

$\Rightarrow$ The statement is true for $n = 1$.

Step II. Assume that the statement is true for $n = k$, i.e. assume 9 is a factor of $5^{2k} + 3k - 1$, so that

$$5^{2k} + 3k - 1 = 9M$$

for some natural number M.

Step III. Consider $n = k + 1$. Then

$$\begin{aligned} 5^{2n} + 3n - 1 &= 5^{2(k+1)} + 3(k+1) - 1 \\ &= 5^2(5^{2k}) + 3k + 2 \\ &= 25[9M - 3k + 1] + 3k + 2, \text{ using the assumption in Step II} \\ &= 25(9M) - 72k + 27 \\ &= 9(25M - 8k + 3), \text{ which is a multiple of 9} \end{aligned}$$

$\Rightarrow$ The statement is true for $n = k + 1$.
Hence, by the principle of mathematical induction, the statement is true for all natural numbers n.

Practice Exercise 4
Prove, by mathematical induction, that for all natural numbers $n = 1, 2, 3, \ldots$

1. $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$

2. $\frac{1}{1(3)}+\frac{1}{3(5)}+\frac{1}{5(7)}+\cdots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$

3. $1^2+2^2+3^2+\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$

4. $1^2+3^2+5^2+\cdots+(2n-1)^2=\frac{1}{3}n(4n^2-1)$

5. $7^{2n}+2$ is an integer multiple of 8

6. 9 divides $2^{2n}-3n+8$

7. 5 divides n^5-n+5

8. 8 is a factor of $7^{2n+1}+1$

9. $7+11+15+\cdots+(4n+5)=2n^2+5n$

10. $3^{n+2}-8n-8$ is an integer multiple of 64.

Summary
We are able to apply the principle of mathematical induction to prove several types of mathematical statements which are true for all natural numbers.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 121-129.

LECTURE FIVE

Permutations and Combinations

Introduction

We shall study the numbers of permutations or arrangements (general or special) of

(a) r objects out of n different objects in a row;

(b) n objects, which are not all different in a row; and

(c) n different objects in a circular arrangement.

We shall also study the number of

(i) ways of choosing r objects in order from n different objects, with or without replacement;

(ii) combinations or selections of r objects out of n different objects with or without replacement or with some other restrictions; and

(iii) partitions of n different objects into classes.

Objectives
The reader should be able to

- use and interpret factorial notation;
- find the number of permutations (general or special) of objects, whether all different or not, in a row or in a circular arrangement;
- find the number of ways of choosing and the number of combinations of r objects from n different objects, with or without replacement or with some other restrictions; and
- find the number of partitions of different objects into classes.

Pre-Test

1. Evaluate (a) 6! (b) ${}^7P_4$ (c) ${}^8C_5$
2. Write in factorial form
   (a) $5 \times 4 \times 3 \times 2 \times 1$
   (b) $7 \times 6 \times 5$
   (c) $n(n-1)(n-2)$
3. Find the number of all the five-digit numbers which can be formed from the digits $1, 2, 3, 4, 5$ assuming that
   (a) a digit cannot be repeated in any number; and
   (b) repetitions are allowed.
4. How many four-letter code-words can be formed form the letters A, B, C, D, E, F assuming that
   (a) a letter cannot be repeated in a code-word; and
   (b) a letter may be repeated?
5. How many permutations can be formed from the word SOTOKO?

6. A committee of 7 are to seat round a circular table in such a way that the Secretary must seat next to the Chairman. In how many ways can the committee be seated?

7. A family of 7 including 2 sisters are to seat on seven adjacent seats at a cinema such that the 2 sisters do not seat together. In how many ways can the family be seated?

8. How many odd numbers can be formed with the digits $2, 3, 4, 5$ and 6 if

   (a) repetitions are not allowed?

   (b) if repetitions are allowed?

9. Suppose that there are 8 balls of different colours in a bag. Four balls are to be drawn from the bag. Find the number of ways of drawing them in order

   (a) allowing replacement for each choice;

   (b) without replacement for each choice.

10. Find the number of combinations of 5 objects from 9 different objects.

11. Two goalkeepers, 6 defenders, 6 mid-fielders and 3 strikers have been invited by a football coach. How many team selections can be make consisting of 1 goalkeeper, 4 defenders, 4 mid-fielders and 2 strikers, (4-4-2 system)?

12. In how many ways can a group of 12 people be divided into three groups of 3, 4 and 5?

Factorial Notation

Denote, in a short notation, the long expression

$$n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

by $n!$ (read as "n factorial").
For example,

$$\begin{aligned} 1! &= 1,\ 2! = 2 \times 1 = 2, 3! = 3 \times 2 \times 1 = 6, \\ 4! &= 4 \times 3 \times 2 \times 1 = 24, \text{etc..} \end{aligned}$$

Example 1
Evaluate:

$$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Example 2
Write in factorial form

$$\begin{aligned}
(a) \qquad & 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! \\
(b) \qquad & 9 \times 8 \times 7 \times 6 \\
= \quad & \frac{9 \times 8 \times 7 \times 6 \times (5 \times 4 \times 3 \times 2 \times 1)}{5 \times 4 \times 3 \times 2 \times 1} \\
= \quad & \frac{9!}{5!} \\
(c) \qquad & r(r-1)(r-2)(r-3) \\
= \quad & \frac{r(r-1)(r-2)(r-3)(r-4)\cdots(2)(1)}{(r-4)\cdots(2)(1)} \\
= \quad & \frac{r!}{(r-4)!} \\
(d) \qquad & n(n-1)(n-2)\cdots(n-r+1) \\
& \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots(2)(1)}{(n-r)\cdots(2)(1)} \\
= \quad & \frac{n!}{(n-r)!}
\end{aligned}$$

Practice Exercise 5A

1. Evaluate
   (a) $5!$, (b) $\frac{8!}{4!}$ (c) $\frac{9!}{3!6!}$

2. Write in factorial form

   (a) $6 \times 5 \times 4 \times 3 \times 2 \times 1$

   (b) $7 \times 6 \times 5$

   (c) $(r-1)(r-2)(r-3)(r-4)$

   (d) $n(n-1)(n-2)\cdots(n-r)$

Permutations
Permutations are arrangements of objects in a row or circle.
Fundamental Principle of Permutations.
If the first position can be filled in r_1 ways, then for each of these ways the second position can be filled in r_2 ways, then the third in r_3 ways and so on up to the k-th position in r_k ways, then the total number of permutations is

$$r_1 \times r_2 \times r_3 \times \cdots \times r_k.$$

Special Cases

1. **Permutations of n objects out of n different objects, denoted ${}^nP_n$.**
   ${}^nP_n = n(n-1)(n-2)\cdots(2)(1)$
   i.e. $\boxed{{}^nP_n = n!}$
   Example 1
   Find the number of all the four-digit numbers which can be formed from the digits 5,6,7,8 assuming that a digit cannot be repeated in any number.
   The number = ${}^4P_4 = 4!$
   $= 4 \times 3 \times 2 \times 11 = 24.$

2. **Permutations of r objects out of n different objects, denoted by ${}^nP_r$**

$$\begin{aligned} {}^nP_r &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n(n-1)(n-2)\cdots(n-r+1)(n-r)\cdots(2)(1)}{(n-r)\cdots(2)(1)} \end{aligned}$$

i.e. $\boxed{{}^nP_r = \dfrac{n!}{(n-r)!}}$

Since ${}^nP_n = n! = \frac{n!}{0!}$, defined $0! = 1$.

Example 2

How many five-letter code-words can be formed from the letters A, B, C,D, E,F,G assuming that a letter cannot be repeated in a code-word?

$$\begin{aligned} \text{The number} &= {}^7P_5 = \frac{7!}{2!} \\ &= 7 \times 6 \times 5 \times 4 \times 3 = 2520 \end{aligned}$$

3. **Permutations of r objects out of n different objects, with repetitions allowed**

$$\begin{aligned} &= n \times n \times \cdots \times n \quad (r \text{ factors}) \\ &= n^r \end{aligned}$$

Example 3

How many four-digit numbers can be formed from the digits 4,5,6,7,8 assuming that repetitions are allowed?

$$\begin{aligned} \text{The number} &= 5 \times 5 \times 5 \times 5 \\ &= 5^4 = 625 \end{aligned}$$

4. **Permutations of objects which are not all different.**

The number of permutations of n objects, out of which r_1 are of one kind, r_2 are of a second kind and so on up to r_k of a final kind, where $r_i > 1$, $1 \leq i \leq k$, and the others are all different is

$$\frac{n!}{r_1!r_2!\cdots r_k!}$$

Example 4

How many permutations can be formed form the word OSOGBO?

Object	O	S	G	B	Total
No. of times	3	1	1	1	6

$$\text{Number of permutations} = \frac{6!}{3!} = 6 \times 5 \times 4 = 120.$$

5. **Circular Arrangement**
   The number of ways of arranging n different objects in a circle is

$$(n-1)!$$

Explanation

In a circular arrangement, we assume that there is no preferred beginning or end, so that any rotation of a circular arrangement does not produce a new arrangement.
Hence, a circular arrangement is equivalent to n permutations in a row. Therefore, number of distinct circular arrangements of n different objects

$$= \frac{n!}{n} = (n-1)!$$

Example 5

A family of 6 are to seat round a circular table. In how many ways can the family be seated if two sisters must (a) sit together, (b) not sit together?

(a) ANALYSE the problem as follows.
Let the two sisters be represented by A and B. If the sisters sit together as AB or BA, they can be considered as one object.
As AB, the family can be seated in $(5-1)! = 4!$ ways $= 24$ ways. As BA, the family can be seated in $(5-1)! = 4!$ ways $= 24$ ways.
Total number of ways

$$= 24 + 24 = 48 \text{ ways.}$$

(b) If it is free sitting the family can be seated in $(6-1)! = 5!$ ways $= 120$ ways.
If the two sisters must not sit together, the family can be seated in $(120 - 48) = 72$ ways.

Example 6

Find the number of even numbers which can be formed from the digits 3, 4, 5, 6 and 7.

(a) if repetitions are not allowed;

(b) if repetitions are allowed.

ANALYSE the problems as follows:
The types of even numbers are

1-digit even numbers

2-digit even numbers

3-digit even numbers

4-digit even numbers

5-digit even numbers.

Note that the numbers can only end with 4 or 6.

(a) If repetitions are not allowed:
Number of 1-digit even numbers = 2
Number of 2-digit even numbers

$$= 4 \times 2 = 8$$

Number of 3-digit even numbers

$$= 4 \times 3 \times 2 = 24$$

Number of 4-digit even numbers

$$= 4 \times 3 \times 2 \times 2 = 48$$

Number of 5-digit even numbers

$$= 4 \times 3 \times 2 \times 1 \times 2 = 48$$

Total number of even numbers

$$= 2 + 8 + 24 + 48 + 48 = 130.$$

(b) If repetitions are allowed:
Number of 1-digit even numbers = 2
Number of 2-digit even numbers

$$= 5 \times 2 = 10$$

Number of 3-digit even numbers

$$= 5 \times 5 \times 2 = 50$$

Number of 4-digit even numbers

$$= 5 \times 5 \times 5 \times 2 = 250$$

Number of 5-digit even numbers

$$5 \times 5 \times 5 \times 5 \times 2 = 1250$$

Total number of even numbers

$$\begin{aligned} &= 2 + 10 + 50 + 250 + 1250 \\ &= 1562. \end{aligned}$$

Practice Exercise 5B

1. Find the number of all five-figure numbers which can be formed from digits 3,4,5,6,7 assuming that a digit
   (a) cannot be repeated
   (b) can be repeated.
2. Find the number of all the four-figure numbers which can be formed from digits 3, 4, 5, 6, 7, 8 assuming that a digit
   (a) cannot be repeated
   (b) can be repeated.

3. How many 4-letter code-words can be formed using the letters P, Q, R, S, T, U, V if repetitions of letters are (a) allowed, (b) not allowed.

4. How many permutations can be formed from the word

(a)	RUNNING	(b)	IBADAN
(c)	LONDON	(d)	ENUGU
(e)	KADUNA	(f)	LAGOS
(g)	ABUJA	(h)	MATHEMATICS
(i)	ZOOLOGY	(j)	DIFFERENTIATION

5. A family of 7 are to seat round a circular table for dinner. In how many ways can the family be seated if

   (a) there is free seating;

   (b) the mother must sit next to the left of the father.

6. Six flags, consisting of one blue, three identical red and two identical green flags, are to be arranged in a row. In how many ways can this be done if

   (a) all six flags are to be used?

   (b) at least 5 of the flags are to be used?

7. How many numbers greater than 6000 can be formed using the digits 4, 5, 6, 7 and 8 if no digit is repeated?

8. How many 3-digit and 4-digit numbers can be formed from the digits 2,3,4,5,6 and 7 if repetitions are not allowed?
   How many are greater than 500?
   How many are odd numbers?

Combinations
Combinations are selections or choices.
Fundamental Principle of Choice
If a first choice can be made in r_1 ways, then for each of these ways, a second choice can be made in r_2 ways, then a third in r_3 ways, and so on up to a

k-th choice being made in r_k ways, then the total number of ways the choices can be made in the given order is

$$r_1 \times r_2 \times r_3 \times \cdots \times r_k$$

Special Cases

(I) Number of ways of choosing r objects in order from n different objects

(a) without replacement
$= n \times (n-1) \times (n-2) \times \cdots \times (n-r+1)$

(b) with replacement
$= n^r$.

(II) The number of combinations of r objects out of n different objects, denoted by ${}^nC_r$ or $\begin{pmatrix} n \\ r \end{pmatrix}$ (read as "n combination r")
$= \dfrac{n!}{r!(n-r)!}$

Explanation
A combination or selection or choice of r objects is equivalent to $r!$ arrangements. Since the total number of arrangements of r objects out of n different objects is $\dfrac{n!}{(n-r)!}$, it follows that the total number of distinct combinations is $\dfrac{1}{r!}\left(\dfrac{n!}{(n-r)!}\right)$.

Example 1.
Evaluate ${}^6C_4$

$$n = 6,\ r = 4$$

$$\begin{aligned} {}^nC_r &= \frac{n!}{r!(n-r)!} \\ \Rightarrow\ {}^6C_4 &= \frac{6!}{4!(6-4)!} = \frac{6 \times 5 \times 4!}{4!2!} \\ &= \frac{30}{2} = 15 \end{aligned}$$

Example 2
There are 9 balls of different colours in a bag. Three balls are to be drawn from the bag. Find the number of ways of drawing them in order

(a) allowing replacement for each choice;

(b) without replacement for each choice.

(a) Number of ways $= 9^3 = 9 \times 9 \times 9 = 729$
(b) Number of ways $= 9 \times 8 \times 7 = 504$

Example 3
Find the number of combinations of 6 objects from 10 different objects

$$\begin{aligned}
\text{The number} &= {}^{10}C_6 = \frac{10!}{6!4!} \\
&= \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!4!} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} \\
&= 210
\end{aligned}$$

(III) If a first choice is made of r_1 objects from n_1 different objects, followed by a second choice of r_2 objects from n_2 different objects, and so on up to the k-th choice of r_k objects from n_k different objects, then the total number of ways the choices can be made

$$= {}^{n_1}C_{r_1} \times {}^{n_2}C_{r_2} \times \cdots \times {}^{n_k}C_{r_k}.$$

Example 4
In how many ways can 2 boys and 3 girls be chosen from a group of 6 boys and 9 girls.?

$$\begin{aligned}
\text{Number of ways} &= {}^6C_2 \times {}^9C_3 \\
&= \frac{6!}{2!4!} \times \frac{9!}{3!6!} \\
&= \frac{6 \times 5}{2 \times 1} \times \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \\
&= 15 \times 84 = 1260
\end{aligned}$$

(IV) **Partitions of Distinct Objects**

The number of partitions of n distinct objects into classes where the first class contains r_1 objects, the second r_2 and so on with the last class containing r_k objects, where $r_1 + r_2 + \cdots + r_k = n$

$$= \frac{n!}{r_1! \times r_1! \times \cdots r_k!}$$

Explanation

The number is obtained from

$$^nC_{r_1} \times \ ^{n-r_1}C_{r_2} \times \cdots \times \ ^mC_{r_k}$$

where $m = n - \sum_{i=1}^{k-1} r_i$.

Example 5

In how many different ways can a group of 8 table-tennis players be partitioned into a first team of 3, a second team of 3 and a reserve pool of 2 players?

$$\begin{aligned} \text{Number of ways} &= \frac{8!}{3!3!2!} \\ &= \frac{8 \times 7 \times 6 \times 5 \times 4}{3 \times 2 \times 1 \times 2 \times 1} = 560 \end{aligned}$$

Practice Exercise 5C

1. Evaluate

   (a) $^9C_4$ (b) $^nC_0$ (c) $^nC_1$
   (d) $^nC_2$ (e) $^nC_{n-2}$ (f) $^nC_{n-1}$
   (g) ^nC_n

2. Show that

   (a) $^nC_r = \ ^nC_{n-r}$

   (b) $^nC_r + \ ^nC_{r+1} = \ ^{n+1}C_{r+1}$

(c) $^nC_r = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots(2)(1)}$

(d) $^nC_r = \dfrac{n(n-1)(n-2)\cdots(r+1)}{(n-r)(n-r-1)\cdots(2)(1)}$

3. There are 8 balls of different colours in a bag. Three balls are to be drawn from the bag. Find the number of ways of drawing them in order

   (a) allowing replacement for each choice;

   (b) without replacement for each choice.

4. A delegation of 3 people is chosen from 5 men and 6 women so that it contains at least 2 women. In how many ways can this be done?

5. In how many ways can a committee of 7 be chosen from 8 men and 7 women so that it contains at least 2 men and at least 3 women?

6. In how many different ways can a group of 10 people be divided into three groups of 5, 3 and 2?

Summary
We are able to
- manipulate fractorial notation.
- determine permutations of objects, whether all different or not, in a row or in a circular arrangement
- calculate the number of ways of choosing as well as the number of combinations of different objects; and
- find the number of partitions of different objects into classes.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 131-147.

LECTURE SIX

The Binomial Theorem

Introduction
The expansion of $(a+b)^n$ is called the binomial theorem. We shall consider the cases where the index n is a positive integer and where n is any real number index.
When n is a positive integer, the expansion is finite and is valid for any real values of a and b.
When n is not a positive integer, the expansion is infinite and the range of values of a and b for which the expansion is valid is limited.

Objectives
The reader should be able to use the binomial theorem to expand $(a+b)^n$ for any positive integral index n, and also for any real number index n.

Pre-Test
1. Use the binomial theorem to expand $(2-x)^8$.
2. Find the coefficient of x^{11} in the expansion of $\left(x^2 + \frac{1}{x}\right)^{10}$.

In Questions 3-6, find the first four terms in the binomial expansion, stating the range of values of x for which each expansion is valid.

3. $\dfrac{1}{(1-x)^2}$ 4. $\dfrac{1}{(1+2x)^7}$

5. $(2-x)^{2/3}$ 6. $\dfrac{1}{\sqrt[4]{1+x}}$

The Binomial Theorem with a Positive Integral Index

If a and b are any real numbers and n is a positive integer, then

$$\begin{aligned}(a+b)^n &= a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \cdots + {}^nC_{n-1}ab^{n-1} + b^n \\ &= \sum_{r=0}^{n} {}^nC_r a^{n-r} b^r\end{aligned}$$

where the ${}^nC_r$ are called binomial coefficients.

Remark.

The sigma notation Σ is used to write the long expression in a compressed form.

Properties

1. Pascal's triangle

$$\begin{array}{lccccccccccccccc}
n=0: & & & & & & & & 1 & & & & & & & \\
n=1: & & & & & & & 1 & & 1 & & & & & & \\
n=2: & & & & & & 1 & & 2 & & 1 & & & & & \\
n=3: & & & & & 1 & & 3 & & 3 & & 1 & & & & \\
n=4: & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
n=5: & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
n=6: & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
n=7: & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}$$

Each number in the triangle, except those at the ends of the rows, which are always equal to 1, is the sum of the two nearest numbers in the row above it.

The numbers in the n-th row of a Pascal's triangle represent the binomial coefficients in the expansion of $(a+b)^n$. For example,

$$\begin{aligned}(a+b)^7 &= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^4 + 35a^3b^4 \\ &\quad +21a^2b^5 + 7ab^6 + b^7\end{aligned}$$

2. The binomial coefficients are symmetrical about the middle.

3. $(a+b)^n$ has $(n+1)$ terms.

4. The $(r+1)$th term, ${}^nC_r a^{n-r} b^r$, is called the general term.

Example 1
Use the binomial theorem to expand $(3-x)^5$.
$a = 3$, $b = -x$, $n = 5$

$$\begin{aligned} (a+b)^n &= a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-1} b^2 + \cdots + {}^nC_{n-1} a b^{n-1} + b^2 \\ \Rightarrow (3-x)^5 &= 3^5 + {}^5C_1 (3^4)(-x) + {}^5C_2 (3^3)(-x)^2 \\ &\quad + {}^5C_3 (3^2)(-x)^3 + {}^5C_4 (3)(-x)^4 + (-x)^5 \\ &= 243 - 405x + 270x^2 - 90x^3 + 15x^4 - x^5. \end{aligned}$$

Example 2
Use the general term to find the coefficient of x^{12} in the binomial expansion of $\left(x^2 - \frac{1}{x}\right)^9$.
$a = x^2$, $b = -\frac{1}{x}$, $n = 9$.
The general term is the

$$\begin{aligned} (r+1)\text{th term} &= {}^nC_r a^{n-r} b^r \\ &= {}^9C_r (x^2)^{9-r} \left(-\frac{1}{x}\right)^r = {}^9C_r x^{18-3r} (-1)^r \\ x^{18-3r} &= x^{12} \Rightarrow 18 - 3r = 12 \Rightarrow r = 2 \end{aligned}$$

$\Rightarrow$ Coefficient of x^{12}, which is the third term

$$= {}^9C_2 (-1)^2 = \frac{9 \times 8}{2 \times 1} = 36$$

Example 3
Find the middle terms in the binomial expansion of $\left(2x^3 - \frac{1}{3x^2}\right)^5$.

There are $(5+1) = 6$ terms in the expansion. The middle terms are therefore the 3rd and 4th terms.

$$a = 2x^3, \ \ b = -\frac{1}{3x^2}, \ \ n = 5$$

The general term, the $(r+1)$th term,

$$= \ {}^nC_r a^{n-r} b^r$$

3rd term (i.e. when $r = 2$)

$$\begin{aligned} &= \ {}^5C_2 (2x^3)^{5-2} \left(-\frac{1}{3x^2}\right)^2 \\ &= \ 10 \times 8x^9 \times \frac{1}{9x^4} = \frac{80}{9}x^5 \end{aligned}$$

4th term (i.e. when $r = 3$)

$$\begin{aligned} &= \ {}^5C_3 (2x^3)^{5-3} \left(-\frac{1}{3x^2}\right)^3 \\ &= \ 10 \times 4x^6 \times \left(-\frac{1}{27x^6}\right) \\ &= \ -\frac{40}{27}, \ \ \text{a constant term.} \end{aligned}$$

Example 4

Find the term not containing x in the binomial expansion of $\left(x^3 - \frac{1}{2x}\right)^8$.
The general term, the $(r+1)$th term

$$\begin{aligned} &= \ {}^nC_r a^{n-r} b^r, \ \ a = x^3, \ \ b = \frac{-1}{2x}, \ n = 8 \\ &= \ {}^8C_r (x^3)^{8-r} \left(-\frac{1}{2x}\right)^r \\ &= \ {}^8C_r x^{24-4r} \left(-\frac{1}{2}\right)^r \end{aligned}$$

$$x^{24-4r} = x^0 \Rightarrow 24 - 4r = 0 \Rightarrow r = 6$$

Hence, the 7-th term is the term not containing x and it is

$$ {}^{8}C_6 \left(-\frac{1}{2}\right)^6 = \frac{8\times 7}{2\times 1} \times \frac{1}{64} = \frac{7}{16}. $$

Practice Exercise 6A

1. Use the binomial theorem to expand

$$ (4+3x)^6 $$

2. Use the general term to find the coefficient of x^2 in the binomial expansion of $\left(\dfrac{x^2}{3} - \dfrac{2}{x^3}\right)^6$

3. Find the middle term in the binomial expansion of $(3-2x)^9$.

4. Find the term not containing x in the expansion of $\left(3x^3 - \dfrac{1}{2x}\right)^{12}$

The Binomial Theorem with any real number index

If q is any real number and $|x| < 1$, then we have the binomial series:

$$ (1+x)^q = 1 + qx + \frac{q(q-1)}{2!}x^2 + \frac{q(q-1)(q-2)}{3!}x^3 + \cdots $$

Remark. If q is a positive integer, the binomial series stops and becomes a polynomial with $(q+1)$ terms. Since we have a finite sum, the condition that $|x| < 1$ is not necessary, and the expansion is valid for any real number x in this case.

Example 1

Find the first four terms in the binomial expansion of $\dfrac{1}{1-x}$ and state the range of validity for x.

$$ \frac{1}{1-x} = (1-x)^{-1}, \; q = -1 $$

$$\begin{aligned}(1+x)^q &= 1+qx+\frac{q(q-1)}{2!}x^2+\frac{q(q-1)(q-2)}{3!}x^3+\cdots\\ \Rightarrow\ (1-x)^{-1} &= 1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2+\frac{(-1)(-2)(-3)}{3!}(-x)^3+\cdots\\ &= 1+x+x^2+x^3+\cdots.\end{aligned}$$

Range of validity is $|-x|<1$ i.e., $|x|<1$.

Example 2

Find the first four terms in the binomial expansion of $(1+2x)^{1/2}$ and state the range of validity of x.

$$\begin{aligned}(1+x)^q &= 1+qx+\frac{q(q-1)}{2!}x^2+\frac{q(q-1)(q-2)}{3!}x^3+\cdots\\ (1+2x)^{1/2} &= 1+\frac{1}{2}(2x)+\frac{\frac{1}{2}(-\frac{1}{2})}{2!}(2x)^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(2x)^3+\cdots\\ &= 1+x-\frac{1}{2}x^2+\frac{1}{2}x^3+\cdots\end{aligned}$$

The range of validity is $|2x|<1$ i.e. $|x|<\frac{1}{2}$.

Example 3

Find the first four terms in the binomial expansion of $\dfrac{1}{\sqrt[3]{2-x}}$ and state the range of values of x for which the expansion is valid.

$$\begin{aligned}\frac{1}{\sqrt[3]{2-x}} &= (2-x)^{-1/3}=\left[2\left(1-\frac{x}{2}\right)\right]^{-1/3}\\ &= 2^{-1/3}\left(1-\frac{x}{2}\right)^{-1/3}=\frac{1}{\sqrt[3]{2}}\left(1-\frac{x}{2}\right)^{-1/3}\\ (1+x)^q &= 1+qx+\frac{q(q-1)}{2!}x^2+\frac{q(q-1)(q-2)}{3!}x^3+\cdots\end{aligned}$$

$$\Rightarrow\ \frac{1}{\sqrt[3]{2}}\left(1-\frac{x}{2}\right)^{\frac{-1}{3}},\quad q=-\frac{1}{3}$$

$$=\ \frac{1}{\sqrt[3]{2}}\left[1+\left(-\frac{1}{3}\right)\left(-\frac{x}{2}\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2!}\left(-\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)\left(-\frac{7}{3}\right)}{3!}\left(-\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{1}{\sqrt[3]{2}}\left(1 + \frac{x}{6} + \frac{1}{18}x^2 + \frac{7}{324}x^3 + \cdots\right)$$

Range of validity is $\left|-\frac{x}{2}\right| < 1$ i.e. $|x| < 2$.

Practice Exercise 6B

Find the first four terms in the binomial expansion, stating the range of values of x for which the expansion is valid.

1.	$(1+x)^{-3}$	2.	$(2-x)^{-2}$
3.	$(1+2x)^{-1}$	4.	$(1-x)^{1/4}$
5.	$(1+x)^{-1/2}$	6.	$(1+2x)^{1/3}$
7.	$(2+x)^{-2/3}$	8.	$(4-3x)^6$

Summary

We are able to use the binomial theorem to expand $(x+b)^x$ for any positive integral index n, and also for any real number index n.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 148-163.

LECTURE SEVEN

Sequences and Series

Introduction
We shall study the following types of sequences and series

- Arithmetical and Geometrical Progressions;
- Those defined by Recurrence Relations;
- Telescoping series; and
- Series of the type $\sum_{k=1}^{n} k^r, r = 0, 1, 2.$

Objectives
The reader should be able to

- calculate the n-th term and the sum of the first n terms of a given Arithmetical and Geometrical Progressions;
- find the n-th term of sequences defined by recurrence relations;
- evaluate telescoping series; and
- use series of the type $\sum_{k=1}^{n} k^r, r = 0, 1, 2$

Pre-Test

1. Find three numbers in an A.P. such that their sum is 27 and their product is 504.

2. Find the n-th term of the sequence:

   (a) $1, 3, 6, 10, 15, \dots$

   (b) $2, 5, 10, 17, 26, \dots$

   (c) $4, 12, 24, 40, 60, \dots$

3. Evaluate $\displaystyle\sum_{k=0}^{n} \frac{7}{10^k}$

4. The numbers $-1, 2, 5$ are the first three terms of an A.P. or a G.P. Which one is it? Find its n-th term and the sum of its first n terms.

5. Find the n-th term of the sequence as a function of n only.

   (a) $S_n = 4n^2 - 9n$

   (b) $a_{n+1} = 2 + 3a_n,\ a_1 = 1$

6. Find the n-th term and the sum of the first n terms

$$\frac{1}{1(3)} + \frac{1}{3(5)} + \frac{1}{5(7)} + \cdots$$

7. Find the sum $\displaystyle\sum_{k=1}^{n} \frac{1}{k(k+3)}$

8. Assuming $\displaystyle\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, determine

   (a) $\displaystyle\sum_{k=1}^{n}(k-2)(2k+4)$ and the sum of the first n terms of the series

   (b) $2 + 5 + 10 + 17 + 26 + \cdots$

(c) $1(3) + 2(5) + 3(7) + \cdots$

Arithmetical Progression (A.P.)
A sequence of the form

$$a, a+d, a+2d, a+3d, \cdots$$

is called an Arithmetical Progression (A.P.) where $a_1 = a$ is the first term and d is the common difference, i.e.

$$a_{k+1} - a_k = d \text{ for all } k,\ 1 \leq k < n$$

The n-th term of an A.P.
$$a_n = a + (n-1)d$$
a = first term, d = common difference.

The sum S_n of the first n terms of an A.P. is called an Arithmetic Series, i.e.

$$S_n = a + (a+d) + \cdots + [a + (n-2)d] + [a + (n-1)d]$$

The sum of the first n terms of an A.P. is
$$S_n = \frac{n}{2}[2a + (n-1)d] = \frac{n}{2}[a_1 + a_n]$$

Example 1
The sum of the first 10 terms of an A.P. is $142\frac{1}{2}$ and the 15-th term is 38. Find the common difference and the sum of the first 21 terms.
For an A.P., if $a_1 = a$
$$S_n = \frac{n}{2}[2a + (n-1)d]$$

When $n = 0$, $s_{10} = 142\frac{1}{2}$, then

$$142\tfrac{1}{2} = 5(2a + 9d)$$

$$\Rightarrow \qquad 2a + 9d = 28.5 \qquad (1)$$

$$a_n = a + (n-1)d$$

When $n = 15$, $a_{15} = 38$, then

$$\begin{array}{lll} & 38 = a + 14d & (2) \\ & 2a + 9d = 28.5 & (1) \\ \underline{(2)\times 2} & \underline{2a + 28d = 76} & \\ \text{Subtract:} & 19d = 47.5 & \end{array}$$

$$d = \frac{47.5}{19} = \frac{95}{2\times 19} = \frac{5}{2}$$

$\Rightarrow$ The common difference $= \dfrac{5}{2}$.

$$\begin{aligned} (2) \quad \Rightarrow \quad a &= 38 - 14d = 38 - 35 = 3 \\ s_n &= \frac{n}{2}[2a + (n-1)d] \end{aligned}$$

When $n = 21$, $a = 3$, $d = \dfrac{5}{2}$, then

$$\begin{aligned} s_{21} &= \frac{21}{2}\left[6 + 20\left(\frac{5}{2}\right)\right] = \frac{21}{2} \times \frac{56}{1} = 21 \times 28 \\ &= 588 \end{aligned}$$

Example 2
Find the n-th term of the sequence

$$2, 7, 15, 26, 40, \ldots$$

Analyse the sequence as follows:

$$\begin{aligned} a_1 &= 2 = 2 \\ a_2 &= 7 = 2 + 5 \\ a_3 &= 15 = 2 + 5 + 8 \\ a_4 &= 26 = 2 + 5 + 8 + 11 \\ a_5 &= 40 = 2 + 5 + 8 + 11 + 14 \end{aligned}$$

Study the pattern above and see that

$$a_n = 2 + 5 + 8 + 11 + 14 + \cdots + y$$

where

$$\begin{aligned}
y &= a_n \text{ of A.P. with } a' = 2,\ d = 3 \\
&= a' + (n-1)d = 2 + (n-1)3 \\
&= 3n - 1. \\
\Rightarrow\ a_n &= 2 + 5 + 8 + 11 + 14 + \cdots + (3n-1) \\
&= s_n \text{ of A.P. with } a'_1 = 2,\ a'_n = 3n - 1 \\
&= \frac{n}{2}(a'_1 + a'_n) \\
&= \frac{n}{2}(2 + 3n - 1) \\
&= \frac{1}{2}n(3n+1).
\end{aligned}$$

Example 3
Find the n-th term of the sequence:

$$2, 6, 13, 23, 36, \ldots$$

Analyse the sequence as follows:

$$\begin{aligned}
a_1 &= 2 = 2 \\
a_2 &= 6 = 2 + 4 \\
a_3 &= 13 = 2 + 4 + 7 \\
a_4 &= 23 = 2 + 4 + 7 + 10 \\
a_5 &= 36 = 2 + 4 + 7 + 10 + 13
\end{aligned}$$

Study the pattern above and see that

$$a_n = 2 + (4 + 7 + 10 + 13 + \cdots + x)$$

where

$$\begin{aligned}
x &= a'_{n-1} \text{ of A.P. with } a' = 4, d = 3 \\
&= a' + (n-2)d = 4 + (n-2)3 \\
&= 3n - 2
\end{aligned}$$

$$
\begin{aligned}
\Rightarrow a_n &= 2 + (s_{n-1} \text{ with } a' = 4,\ a'_{n-1} = 3n - 2) \\
&= 2 + \frac{(n-1)}{2}(a' + a'_{n-1}) \\
&= 2 + \frac{(n-1)}{2}(4 + 3n - 2) \\
&= 2 + \frac{(n-1)}{2}(3n + 2) \\
&= \frac{1}{2}(4 + 3n^2 - n - 2) \\
&= \frac{1}{2}(3n^2 - n + 2).
\end{aligned}
$$

Practice Exercise 7A

1. The 3rd and 7th terms of an A.P. are respectively -1 and 11. Find the n-th term and the number of terms which must be taken to get a sum of 430.

2. Find the n-th term of the sequence

   (a) $3, 8, 15, 24, 35, \ldots$

   (b) $3, 10, 21, 36, 55, \ldots$

   (c) $3, 18, 45, 84, 135, \ldots$

   (d) $4, 8, 15, 25, 38, \ldots$

Geometrical Progression (G.P.)
A sequence of the form

$$a, ar, ar^2, ar^3, \ldots$$

is called a Geometrical Progression (G.P.) where $a_1 = a$ is the first term and r is the common ratio; i.e. the ratio $\dfrac{a_{k+1}}{a_k} = r$ for all $k,\ 1 \le k < n$.

The n-th term of a G.P.

$$a_n = ar^{n-1}$$

a = **first term,** r = **common ratio**

The sum s_n of the first n terms of a G.P. is called a Geometric Series.

$$\boxed{\begin{array}{l}\textbf{The sum of the first } n \textbf{ term of a G.P.}\\ \qquad s_n = \dfrac{a(1-r^n)}{1-r}\\ a = \textbf{first term, } r = \textbf{common ratio}\end{array}}$$

Example 1

Evaluate $\sum_{k=1}^{n} \frac{3}{2^k}$.

The series can be expanded as

$$\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \ldots$$

This is a G.P. with $a = \frac{3}{2}$, $r = \frac{1}{2}$

$$\begin{aligned} S_n &= \sum_{k=1}^{n} \frac{3}{2^k} = \frac{a(1-r^n)}{1-r} \\ &= \frac{3}{2}\frac{\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 3\left[1-\frac{1}{2^n}\right] \end{aligned}$$

Example 2

Given that 1, $\sin y$ and $\cos^2 y$ are three consecutive terms of a G.P., find the value of the common ratio.

The common ratio

$$\begin{aligned} r &= \frac{\sin y}{1} \text{ or } r = \frac{\cos^2 y}{\sin y} \\ \Rightarrow \frac{\cos^2 y}{\sin y} &= \frac{\sin y}{1} \Rightarrow \cos^2 y = \sin^2 y \\ \Rightarrow \cos y &= \pm \sin y \Rightarrow \tan y = \pm 1, \\ \Rightarrow r &= \sin y = \pm \frac{1}{\sqrt{2}} \end{aligned}$$

Example 3

Find the n-th term of the series:

$$7 + 77 + 777 + 7777 + \cdots$$

Analyse the sequence as follows:

$$\begin{aligned}
a_1 &= 7 = 7 \\
a_2 &= 77 = 7 + 7(10) \\
a_3 &= 777 = 7 + 7(10) + 7(10^2) \\
a_4 &= 7777 = 7 + 7(10) + 7(10^2) + 7(10^3)
\end{aligned}$$

From the pattern above, we have a G.P. with a_k having k terms, and so

$$\begin{aligned}
a_n &= 7 + 7(10) + 7(10^2) + \cdots + 7(10^{n-1}) \\
&= a\frac{1-r^n}{1-r}, \ a = 7, \ r = 10 \\
&= \frac{7(1-10^n)}{1-10} = \frac{7}{9}(10^n - 1)
\end{aligned}$$

Practice Exercise 7B

1. 1.] Find the n-th term of a G.P. whose 3rd term is 36 and whose 5-th term is 324.

2. If the sum of the first n terms of a sequence is given by

$$s_n = 9\left(1 - \frac{1}{3^n}\right)$$

   (i) find the first and second terms of the sequence;

   (ii) find the n-th term of the sequence;

   (iii) show that the sequence is a G.P. and find the common ratio.

3. Evaluate

   (a) $\displaystyle\sum_{k=2}^{n} \frac{1}{3^k}$

   (b) $\displaystyle\sum_{k=1}^{n} \left(\frac{5}{6}\right)^k$

4. Find the n-th term of a G.P. if its 2nd term is 6 and its 4th term is 54.

5. If x, y, z are in G.P., show that $\log x$, $\log y$ and $\log z$ are in A.P.

6. A house appreciates by 5% every year. If the value of the house, when new, is Y dollars, find its value after 5 years.

7. Find the n-th term of the series

   (a) $4 + 44 + 444 + 4444 + \cdots$

   (b) $5 + 55 + 555 + 5555 + \cdots$

Sequences Defined by Recurrence Relations
If we have a recurrence, in which a_{n+1} is given in terms of the previous term, we wish to find a_n in terms of n only. We use the recurrence relation to obtain the first few terms of the sequence. A study of the pattern formed by the sequence leads to trial expressions for a_n, which must be tested to satisfy the recurrence relation.

Example 1
Find the n-th term of the sequence as a function of n only.

$$a_{n+1} = 3a_n + 4, a_1 = 4$$

$$\begin{aligned}
a_1 = 4 &= 4 = 4(1) \\
a_2 = 3a_1 + 4 &= 16 = 4(1+3) \\
a_3 = 3a_2 + 4 &= 52 = 4(1+3+3^2) \\
a_4 = 3a_3 + 4 &= 160 = 4(1+3+3^3+3^3) \\
a_5 = 3a_4 + 4 &= 484 = 4(1+3+3^2+3^3+3^4)
\end{aligned}$$

From the pattern, try

$$\begin{aligned}
a_n &= 4(1+3+3^2+\cdots+3^{n-1}) \\
&= 4\frac{3^n-1}{3-1} = 2(3^n-1).
\end{aligned}$$

Test and check that the recurrence relation is satisfied.

$$\begin{aligned} L.H.S. &= a_{n+1} = 2(3^{n+1} - 1) \\ R.H.S. &= 3[2(3^n - 1)] + 4 \\ &= 6(3^n) - 6 + 4 = 2(3^{n+1} - 1) = L.H.S. \end{aligned}$$

Hence $a_n = 2(3^n - 1)$.

Example 2
Find the n-th term of the sequence in terms of n only.
$a_{n+1} = a_n + 2n$, $a_1 = 1$

$$\begin{aligned} a_1 &= 1 &&= 1 \\ a_2 &= a_1 + 2(1) &&= 3 &&= 1 + 2 \\ a_3 &= a_2 + 2(2) &&= 7 &&= 1 + 2 + 4 \\ a_4 &= a_3 + 2(3) &&= 13 &&= 1 + 2 + 4 + 6 \\ a_5 &= a_4 + 2(4) &&= 21 &&= 1 + 2 + 4 + 6 + 8 \end{aligned}$$

From the pattern, try

$$\begin{aligned} a_n &= 1 + (S_{n-1} \text{ of A.P. with } a = 2, d = 2) \\ &= 1 + \frac{1}{2}(n-1)[2a + (n-2)d] \\ &= 1 + \frac{1}{2}(n-1)(4 + 2n - 4) \\ &= n^2 - n + 1 \end{aligned}$$

Test and check that the recurrence relation is satisfied

$$\begin{aligned} L.H.S. = a_{n+1} &= (n+1)^2 - (n+1) + 1 \\ &= n^2 + n + 1 \end{aligned}$$

$$\begin{aligned} R.H.S. &= a_n + 2n \\ &= n^2 - n + 1 + 2n = n^2 + n + 1 = L.H.S. \end{aligned}$$

Hence $a_n = n^2 - n + 1$.

Practice Exercise 7C
Find the n-th term of the sequence as a function of n only:

1. $a_{n+1} = 2a_n,\ a_3 = 7$
2. $a_{n+1} = a_n + 8n,\ a_1 = 1$
3. $a_{n+1} = 2a_n + 3,\ a_1 = 3$
4. $a_{n+1} = \dfrac{a_n}{a_n + 1},\ a_1 = 1$

Telescoping Series (Using Partial Fractions)
Example 1.
Find the n-th term and the sum of the first n terms of the series:

$$\frac{1}{2(5)} + \frac{1}{5(8)} + \frac{1}{8(11)} + \cdots$$

The first factors are

$$2, 5, 8, \ldots$$

This is A.P. with $a = 2$, $d = 3$

$$\begin{aligned} n\text{-th term} &= a + (n-1)d \\ &= 2 + (n-1)3 = 3n - 1 \end{aligned}$$

The second factors are

$$5, 8, 11, \ldots$$

This is A.P. with $a = 5$, $d = 3$.

$$\begin{aligned} n\text{-th term} &= a + (n-1)d \\ &= 5 + (n-1)3 = 3n + 2 \end{aligned}$$

Hence the n-th term of the series is

$$\frac{1}{(3n-1)(3n+2)}$$

The first n terms of the series can be expressed, using the sigma notation, as

$$\sum_{k=1}^{n} \frac{1}{(3k-1)(3k+2)}$$

Resolve into a sum of partial fractions:

$$\begin{aligned}\frac{1}{(3k-1)(3k+2)} &= \frac{A}{3k-1}+\frac{B}{3k+2}\\ &= \frac{A(3k+2)+B(3k-1)}{(3k-1)(3k+2)}\end{aligned}$$

$\Rightarrow \quad 1 \equiv A(3k+2)+B(3k-1)$

Put $\underline{3k=1}$: $1 = A(1+2)+0 \Rightarrow A = \dfrac{1}{3}$

Put $\underline{3k=-2}$: $1 = 0 + B(-2-1) \Rightarrow B = -\dfrac{1}{3}$

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{(3k-1)(3k+2)} &= \sum_{k=1}^{n}\left[\frac{1/3}{3k-1}-\frac{1/3}{3k+2}\right]\\ &= \left(\frac{1/3}{2}-\frac{1/3}{5}\right) && k=1\\ &\quad+\left(\frac{1/3}{5}-\frac{1/3}{8}\right) && k=2\\ &\quad+\left(\frac{1/3}{8}-\frac{1/3}{11}\right) && k=3\\ &\quad\cdots\\ &\quad+\left(\frac{1/3}{3n-4}-\frac{1/3}{3n-1}\right) && k=n-1\\ &\quad+\left(\frac{1/3}{3n-1}-\frac{1/3}{3n+2}\right) && k=n\\ &= \frac{1/3}{2}-\frac{1/3}{3n+2}\end{aligned}$$

(All the terms cancel out in pairs except the first and the last terms)

$$\begin{aligned}&= \frac{1}{6}-\frac{1}{3(3n+2)} = \frac{3n+2-2}{6(3n+2)}\\ &= \frac{3n}{6(3n+2)} = \frac{n}{2(3n+2)}.\end{aligned}$$

Example 2
Find the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+2)}$$

Resolve into a sum of partial fractions

$$\begin{aligned}\frac{1}{k(k+2)} &= \frac{A}{k} + \frac{B}{k+2} \\ &= \frac{A(k+2)+Bk}{k(k+2)}\end{aligned}$$

$\Rightarrow \quad 1 \equiv A(k+2) + Bk$
<u>Put $k=0$</u>: $1 = A(0+2) + 0 \Rightarrow A = \dfrac{1}{2}$
<u>Put $k=-2$</u>: $1 = 0 + B(-2) \Rightarrow B = -\dfrac{1}{2}$

$$\begin{aligned}\sum_{k=1}^{n} \frac{1}{k(k+2)} &= \sum_{k=1}^{n} \left[\frac{1/2}{k} - \frac{1/2}{k+2}\right] \\ &= \left(\frac{1/2}{1} - \frac{1/2}{3}\right) && k=1 \\ &\quad + \left(\frac{1/2}{2} - \frac{1/2}{4}\right) && k=2 \\ &\quad + \left(\frac{1/2}{3} - \frac{1/2}{5}\right) && k=3 \\ &\quad + \left(\frac{1/2}{4} - \frac{1/2}{6}\right) && k=4 \\ &\qquad \cdots \\ &\quad + \left(\frac{1/2}{n-2} - \frac{1/2}{n}\right) && k=n-2 \\ &\quad + \left(\frac{1/2}{n-1} - \frac{1/2}{n+1}\right) && k=n-1 \\ &\quad + \left(\frac{1/2}{n} - \frac{1/2}{n+2}\right) && k=n\end{aligned}$$

$$= \frac{1/2}{1} + \frac{1/2}{2} - \frac{1/2}{n+1} - \frac{1/2}{n+2}$$

(All the terms cancel out in pairs except the first 2 positive and the last 2 negative terms)

$$\begin{aligned}
&= \frac{1}{2} + \frac{1}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} \\
&= \frac{3(n+1)(n+2) - 2(n+2) - 2(n+1)}{4(n+1)(n+2)} \\
&= \frac{3n^2 + 5n}{4(n+1)(n+2)} \text{ or } \frac{n(3n+5)}{4(n+1)(n+2)}
\end{aligned}$$

Practice Exercise 7D

1. Find the n-th term and the sum of the first n terms of the series:

$$\frac{1}{3(4)} + \frac{1}{4(5)} + \frac{1}{5(6)} + \cdots$$

2. Find the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)}$$

Use of the Series $\sum_{k=1}^{n} k^r$ $r = 0, 1, 2$

$$\begin{aligned}
\text{I.} \quad \sum_{k=1}^{n} 2 &= 2 + 2 + \cdots + 2 \ (n \text{ times}) \\
&= 2n \\
\text{II.} \quad \sum_{k=1}^{n} k &= 1 + 2 + 3 + \cdots + n \\
&= \frac{1}{2}n(n+1), \quad (\text{A.P. with } a = 1,\ d = 1) \\
\text{III.} \quad \sum_{k=1}^{n} k^2 &= \frac{n}{6}(n+1)(2^n+1)
\end{aligned}$$

(This can be shown by mathematical induction)

Example 1
Assuming $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, find the sum $\sum_{k=1}^{n}(2k-1)(2-k)$

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1)(2-k) &= \sum_{k=1}^{n}(5k-2-2k^2) \\
&= 5\sum_{k=1}^{n} k - \sum_{k=1}^{n} 2 - 2\sum_{k=1}^{n} k^2 \\
&= \frac{5n(n+1)}{2} - 2n - \frac{1}{3}n(n+1)(2n+1), \\
&\quad \text{using the assumption} \\
&= \frac{n}{6}[15(n+1) - 12 - 2(n+1)(2n+1)] \\
&= \frac{n}{6}[15n + 15 - 12 - 2(2n^2 + 3n + 1)] \\
&= \frac{n}{6}(1 + 9n - 4n^2)
\end{aligned}$$

Example 2
Assuming $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, determine the sum of the first n terms of the series.

$$1(3) + 3(6) + 5(9) + \cdots$$

The first n terms of the series can be expressed as

$$\begin{aligned}
\sum_{k=1}^{n}(2k-1)3k &= 6\sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k \\
&= n(n+1)(2n+1) - \frac{3}{2}n(n+1), \\
&\quad \text{using the assumption} \\
&= \frac{1}{2}n(n+1)[2(2n+1) - 3] \\
&= \frac{1}{2}n(n+1)(4n-1)
\end{aligned}$$

Example 3

Assuming $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, determine the sum of the first n terms of the series

$$3 + 8 + 15 + 24 + 35 + \cdots$$

First find the n-th term of the series by analysing the terms:

$$\begin{aligned}
a_1 &= 3 = 2^2 - 1 \\
a_2 &= 8 = 3^2 - 1 \\
a_3 &= 15 = 4^2 - 1 \\
a_4 &= 24 = 5^2 - 1 \\
a_5 &= 35 = 6^2 - 1
\end{aligned}$$

From the pattern, $a_n = (n+1)^2 - 1 = n^2 + 2n$.
The first n terms of the series can be expressed, using the sigma notation, as

$$\begin{aligned}
\sum_{k=1}^{n}(k^2 + 2k) &= \sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k \\
&= \frac{1}{6}n(n+1)(2n+1) + 2 \times \frac{1}{2}n(n+1), \\
&\quad \text{using the assumption} \\
&= \frac{1}{6}n(n+1)[2n+1+6] \\
&= \frac{1}{6}n(n+1)(2n+7)
\end{aligned}$$

Practice Exercise 7E

Assuming $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$, determine the sum of the first n terms of the series:

1. $\sum_{k=1}^{n}(3-2k)(3k+4)$

2. $3 + 10 + 21 + 36 + 55 + \cdots$

3. $2 + 5 + 12 + 23 + 38 + \cdots$

4. $1(3) + (2)(4) + 3(5) + \cdots$

Summary
We are able to determine the n-th terms and the sum of the first n terms of A.P., G.P., sequences defined by recurrence relations, and telescoping series. We are also able to use series of the type
$\sum_{k=1}^{n} k^n, \ r = 0, 1, 2$

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 169-185.

LECTURE EIGHT

Complex Numbers

Introduction

Complex numbers were developed in order to solve equations such as

$$x^2 + 4 = 0$$

which do not have real number solutions.

$$x^2 + 4 = 0 \Rightarrow x^2 = -4$$

$$\Rightarrow \quad x = \pm\sqrt{-4} = \pm\sqrt{4} \cdot \sqrt{-1} = \pm 2\sqrt{-1}.$$

The set of complex numbers, denoted by $\mathbb{C}$, is therefore defined as

$$\mathbb{C} = \{a + ib | a \text{ and } b \text{ are real, and } i^2 = -1\}$$

The complex number $i = \sqrt{-1}$ is called an imaginary number. In a complex number $a + ib$, a is called the real part, while ib is called the imaginary part. The complex numbers include all the real numbers, $a + i0 = a$.

Objectives

The reader should be able to

- add, subtract, multiply and divide complex numbers using the Cartesian form;
- solve polynomial equations with real number coefficients;
- write a complex number in polar form, using the Argand diagram;
- multiply and divide complex numbers, using the polar form; and
- apply de-Moivre's theorem to calculate powers and n-th roots of complex numbers, and to express sine and cosine of multiple angles in terms of powers of sine and cosine of single angles, and vice-versa.

Pre-Test

1. Simplify and express in the form $a + ib$

   (a) $(4 + 3i)(2 + 6i)$

   (b) $\dfrac{2 - 3i}{3 + 2i}$

2. Find all the complex number solutions in the form $a+ib$ of the equations

   (a) $x^2 + 6x + 18 = 0$

   (b) $x^3 - 4x^2 + x + 6 = 0$

3. Find all the complex roots of the equation

$$x^4 - 4x^3 + 14x^2 - 4x + 13 = 0$$

   given that one of the roots is $2 - 3i$.

4. Write in polar form
   (a) $4i - 4$ (b) $-\sqrt{3} - 2i$.

5. Express in Cartesian form the complex number
   (a) $(\sqrt{2}, 45^0)$ (b) $\left(1, -\frac{\pi^c}{2}\right)$

6. Describe the locus of a complex number z which satisfies

$$|z-2| = 3.$$

7. Write in polar form
   (a) $z_1 \cdot z_2$ (b) $\frac{z_1}{z_2}$
   where

$$\begin{aligned} z_1 &= 3(\cos 40^0 + i\sin 40^0) \\ z_2 &= 4(\cos 100^0 + i\sin 100^0) \end{aligned}$$

8. Use de-Moivre's theorem to simplify
   (a) $(1+i)^{10}$ (b) $(1-i)^{16}$
   (c) $(\cos 60^0 + i\sin 60^0)^{15}$

9. Find all the complex fourth roots of
   (a) 16 (b) i (c) $125(3-4i)$

10. (a) Express $\cos 5\theta$ in terms of powers of $\cos\theta$ only.
    (b) Express $\cos^5\theta$ in terms of sine and cosine of multiple angles.

Cartesian Form
The complex number $z = a + ib$ is said to be in Cartesian form.

Addition and Subtraction in Cartesian form
Add or subtract the real parts and the imaginary parts.

Example 1. Simplify

(a) $(3-7i) + (-7+9i) = (3-7) + (-7i+9i) = -4+2i$

(b) $(-4-5i) - (6-7i) = (-4-6) + (-5i+7i) = -10+2i$

Multiplication in Cartesian form
Multiply as in real numbers, but remember
$i = \sqrt{-1}$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$.

Example 2.

(a) $(2-3i)(3+2i) = 6 + 4i - 9i - 6i^2 = 6 - 5i + 6 = 12 - 5i$

(b) $i^7 = i^4 \cdot i^3 = 1(-i) = -i$

Division in Cartesian form
If $z = a + ib$ is any complex number, then $\bar{z} = a - ib$ is called its complex conjugate.

$$\begin{aligned} z \cdot \bar{z} &= (a+ib)(a-ib) = a^2 - iab + iab - i^2b^2 \\ &= a^2 + b^2 \end{aligned}$$

$\boxed{\textbf{The product of a complex number and its conjugate is always a real number.}}$

Use complex conjugates to divide.

Example 3
Simplify and write in the form $a + ib$

$$\begin{aligned} \frac{4-i}{3+2i} &= \frac{4-i}{3+2i} \times \frac{3-2i}{3-2i} \\ &= \frac{12 - 8i - 3i + 2i^2}{3^2 + 2^2} = \frac{10 - 11i}{13} \\ &= \frac{10}{13} - \frac{11}{13}i \end{aligned}$$

Practice Exercise 8A
Simplify and write in the form $a + ib$.

1.	$(2+4i) + (1-8i)$	2.	$(5-2i) - (7+2i)$
3.	$(2+3i)(4-5i)$	4.	$i^6(3-i)$
5.	$(1-3i)^{-1}$	6.	$\dfrac{4+3i}{2+6i}$

Polynomial Equations with real number coefficients
Theorem
If a complex number u is a solution of the polynomial equation

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with real number coefficients, then its complex conjugate $\bar{u}$ is also a solution.

Example 1
Find all the solutions in the form $a + ib$:

$$x^2 + 4x + 9 = 0$$

Use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 4$, $c = 9$. The solutions are

$$\begin{aligned} x &= \frac{-4 \pm \sqrt{16 - 36}}{2} \\ &= -2 \pm \frac{1}{2}\sqrt{-20} = -2 \pm \frac{1}{2}\sqrt{20} \cdot \sqrt{-1} \\ &= -2 \pm i\sqrt{5} \end{aligned}$$

Note: The solutions are conjugates of each other.

Example 2
Solve the equation $x^2 + 12 = 0$

$$\begin{aligned} & x^2 - (\sqrt{-12})^2 = 0 \\ \Rightarrow \quad & x^2 - (i\sqrt{12})^2 = 0 \\ \Rightarrow \quad & (x + i\sqrt{12})(x - i\sqrt{12}) = 0, \\ & \text{(Difference of 2 squares)} \\ \Rightarrow x \quad &= \pm i\sqrt{12} \\ &= \pm 2i\sqrt{3} \end{aligned}$$

Example 3
Find the roots of the polynomial equation

$$x^4 + x^3 + 5x^2 + 4x + 4 = 0$$

given that one of the roots is $2i$.
The conjugate of $2i$, which is $-2i$, is also a solution. Hence

$$(x+2i)(x-2i) = x^2 + 4$$

is a factor. Use the division algorithm to divide:

$$\begin{array}{rl} & \underline{x^2 + x + 1 \qquad} \\ \underline{x^2+4} & | x^4 + x^3 + 5x^2 + 4x + 4 \\ & -(x^4 + \quad 4x^2) \\ & \quad x^3 + x^2 + 4x + 4 \\ & \underline{-(x^3 + \qquad 4x)} \\ & \qquad x^2 + 4 \\ & \qquad \underline{x^2 + 4} \end{array}$$

Equation becomes

$$(x^2+4)(x^2+x+1) = 0$$

The other solutions are solutions of the quadratic equation $x^2 + x + 1 = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = 1$, $c = 1$.

$$\begin{aligned} x &= \frac{-1 \pm \sqrt{1-4}}{2} \\ &= -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \end{aligned}$$

All the 4 roots are

$$2i,\ -2i,\ -\frac{1}{2} + \frac{i\sqrt{3}}{2},\ -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

Practice Exercise 8B
Find all the roots in the form $a + ib$

1. $4x^2 + 5x + 2 = 0$
2. $x^2 + 6 = 0$
3. $x^4 + 4x^3 + 6x^2 + 4x + 5 = 0$, if one of the roots is $x = i$.

Polar Form and the Argand Diagram
A complex number $z = x + iy$ can be represented in an Argand diagram using the coordinate plane

The x-axis is called the Real axis, while the y-axis is called the Imaginary axis. The Argand diagram is also called the Argand plane or the Complex plane.
The length OP is called the **modulus** of the complex number z, where

$$r = |z| = |x + iy| = \sqrt{x^2 + y^2}$$

Note that $z \cdot \bar{z} = |z|^2$.
The angle θ which the line OP makes with the Real axis is called argument or amplitude of the complex number z, and is denoted as arg$z$.

Since on the circle, angles θ and $\theta + 2\pi n$ for any integer n, represent the same angle, it follows that argument of a complex number is not unique.
i.e. $arg\ z = \theta + 2\pi n$, for any integer n.
However, the argument which lies between $-\pi$ and π is called the principal argument, and is denoted $Arg\ z$, with capital A, i.e. $-\pi < Arg\ z \leq \pi$.
From the Argand diagram, given r and θ

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

so that $x = r(\cos\theta + i\sin\theta)$ and given x and y,

$$r^2 = x^2 + y^2 \text{ and } \theta = arg\ z = \tan^{-1}\left(\frac{y}{x}\right)$$

The pair (r, θ) is called the **polar form** representation of the complex number z. It is also called the modulus argument form or the trigonometric form, i.e.

$$(r, \theta) = r(\cos\theta + i\sin\theta)$$

Example 1

Write in polar form

(a) $\sqrt{2} - i$ (b) $-1 - i\sqrt{3}$

Sketch an Argand diagram to locate the complex number

(a)

From the diagram,

$\sqrt{3} - i = (r, -\theta)$

where $r^2 = (\sqrt{3})^2 + (-1)^2 = 4$

$r = 2$

$\tan\theta = \frac{1}{\sqrt{3}},\ \theta = 30^0$

Hence

$$\sqrt{3} - i = (2, -30^0) \text{ or } 2(\cos(-30^0) + i\sin(-30^0))$$

or

$$\sqrt{3} - i = (2, 330^0) \text{ or } 2(\cos 330^0 + i\sin 330^0)$$

(b)

From the diagram,

$-1 - i\sqrt{3} = (r, -180^0 + \theta)$

where $r^2 = (-\sqrt{3})^2 + (-1)^2 = 4$

$r = 2$

$\tan\theta = \sqrt{3},\ \theta = 60^0$

Hence

$$-1 - i\sqrt{3} = (2, -120^0) \text{ or } 2(\cos(-120^0) + i\sin(-120^0))$$

or

$$-1 - i\sqrt{3} = (2, 240^0) \text{ or } 2(\cos 240^0 + i\sin 240^0)$$

Example 2
Write in Cartesian form
(a) $(4, 30^0)$ (b) $(3, 120^0)$

$$\begin{aligned}
\text{(a)}\quad (4, 30^0) &= 4(\cos 30^0 + i \sin 30^0) \\
&= 4\left[\left(\frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2}\right)\right] \\
&= 2\sqrt{3} + 2i \\
\text{(b)}\quad (3, 120^0) &= 3(\cos 120^0 + i \sin 120^0) \\
&= 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) \\
&= -\frac{3}{2} + \frac{3\sqrt{3}}{2}i
\end{aligned}$$

Example 3
Describe the locus of a complex number z which satisfies

$$|2z + 1 - 2i| = 3.$$

Put $z = x + iy$. Then

$$\begin{aligned}
&2x + 1 - 2i = 2x + 2iy + 1 - 2i \\
&= (2x + 1) + (2y - 2)i \\
&|2z + 1 - 2i| = 3 \\
&\Rightarrow\ |2z + 1 - 2i|^2 = 3^2 \\
&\Rightarrow\ (2x + 1)^2 + (2y - 2)^2 = 9 \\
&\Rightarrow\ 4(x + \tfrac{1}{2})^2 + 4(y - 1)^2 = 9 \\
&\Rightarrow\ (x + \frac{1}{2})^2 + (y - 1)^2 = \left(\frac{3}{2}\right)^2
\end{aligned}$$

The locus is a circle, centre at $(-\frac{1}{2}, 1)$ and radius $\frac{3}{2}$ units.

Practice Exercise 8C

1. Write in polar form
   (a) $2 + 2i$ (b) $-2 + 2i\sqrt{3}$
   (c) $-\sqrt{3} - i$ (d) $4 - 4\sqrt{3}i$

2. Write in Cartesian form
   (a) $(2, 60^0)$ (b) $(5, 145^0)$
   (c) $(3, 210^0)$ (d) $(4, -45^0)$

3. Describe the locus of a complex number z which satisfies:

   (a) $|z - 2 + 4i| = 3|z|$

   (b) $|z - 2| = 3|z + 2i|$

Multiplication and Division in Polar Form
It is easier to multiply and divide in polar form than in the Cartesian form.
If $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$

$$\begin{aligned} z_1 \cdot z_2 &= (r_1 r_2,\ \theta_1 + \theta_2) \\ \frac{z_1}{z_2} &= \left(\frac{r_1}{r_2},\ \theta_1 - \theta_2\right) \end{aligned}$$

To multiply, just multiply the moduli and add the arguments.
To divide, just divide the moduli and subtract the arguments.

Example 1
Write in polar form $z_1 \cdot z_2$, where

$$\begin{aligned} z_1 &= 2(\cos 50^0 + i \sin 50^0), \\ z_2 &= 5(\cos 110^0 + i \sin 110^0) \\ z_1 &= (2, 50^0), \\ z_2 &= (5, 110^0) \\ z_1 \cdot z_2 &= (2 \times 5,\ 50^0 + 110^0) = (10, 160^0) \\ &= 10(\cos 160^0 + i \sin 160^0). \end{aligned}$$

Example 2
Write in polar form $\frac{z_1}{z_2}$, where

$$\begin{aligned} z_1 &= 6(\cos 70^0 + i \sin 70^0),\ z_2 = 3(\cos 40^0 + i \sin 40^0) \\ z_1 &= (6, 70^0),\ z_2 = (3, 40^0) \end{aligned}$$

$$\begin{aligned}\frac{z_1}{z_2} &= \left(\frac{6}{3},\ 70^0 - 40^0\right) = (2, 30^0)\\ &= 2(\cos 30^0 + i\sin 30^0).\end{aligned}$$

Practice Exercise 8D

1. Write in polar form $z_1 \cdot z_2$, where
   (a) $z_1 = (3, 40^0), \quad z_2 = (4, 80^0)$
   (b) $z_1 = \cos 100^0 + i\sin 100^0, \quad z_2 = 7(\cos 25^0 + i\sin 25^0)$
2. Write in polar form $\dfrac{z_1}{z_2}$, where
   (a) $z_1 = (4, 80^0), \quad z_2 = (1, 20^0)$
   (b) $z_1 = 9(\cos 125^0 + i\sin 125^0),\ z_2 = 3(\cos 75^0 + i\sin 75^0)$

De-Moivre's Theorem
If $z = r(\cos\theta + i\sin\theta)$ is a complex number expressed in polar form, then

$$\begin{aligned} z^n &= r^n(\cos n\theta + i\sin n\theta)\\ z^{-n} &= r^{-n}[\cos(-n)\theta + i\sin(-n)\theta]\end{aligned}$$

Example 1.
Use de-Moivre's theorem to simplify $\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)^{13}$.

First express $\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ in polar form by sketching an Argand diagram to locate the complex number.

From the diagram
$\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = (r, \theta)$
where $r^2 = \left(\dfrac{1}{2}\right)^2 + \left(\dfrac{\sqrt{3}}{2}\right)^2 = 1,\ r = 1$
$\tan = \sqrt{3},\ \theta = 60^0$.

Hence $\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}=(1,60^0)=(\cos 60^0+i\sin 60^0)$..

$$\begin{aligned}\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{13} &= (\cos 60^0+i\sin 60^0)^{13}\\ &= \cos(13\times 60)^0+i\sin(13\times 60)^0\\ &= (\cos 780^0+i\sin 780^0)\\ &= (\cos 60^0+i\sin 60^0)\text{ or }\frac{1}{2}+\frac{i\sqrt{3}}{2}\end{aligned}$$

Example 2
Use de-Moivres theorem to simplify

$$(\cos 70^0+i\sin 70^0)^{-6}$$

$$\begin{aligned}(\cos 70^0+i\sin 70^0)^{-6} &= \cos(-6\times 70)^0+i\sin(-6\times 70)^0\\ &= \cos(-420^0)+i\sin(-420^0)\\ &= \cos(-60^0)+i\sin(-60^0)\text{ or }\frac{1}{2}-\frac{\sqrt{3}}{2}i\end{aligned}$$

Practice Exercise 8E
Use de-Moivre's theorem to simplify:

1.	$(2-i\sqrt{3})^{12}$	2.	$(\cos 50^0+i\sin 50^0)^{10}$
3.	$(2,35^0)^8$	4.	$(2+i)^6$

Roots of complex numbers
We use de-Moivre's theorem to find the n-th roots of any complex number, α. Then

$$z=\alpha^{1/n}\Rightarrow z^n-\alpha=0$$

By the Fundamental Theorem of Algebra, every polynomial of degree n has n complex number solutions.
Therefore, the n-th root z of α has n values which are the solutions of the polynomial equation

$$z^n-\alpha=0.$$

$$\begin{aligned} \text{If } \alpha &= r(\cos\theta + i\sin\theta) \text{ then} \\ \alpha &= r[\cos(\theta + 360k) + i\sin(\theta + 360k)] \end{aligned}$$

for any integer k. By de-Moivre's theorem, the n-th roots of α are

$$z_k = \alpha^{1/n} = r^{1/n}\left(\cos\frac{\theta + 360k}{n} + i\sin\frac{\theta + 360k}{n}\right), \quad k = 0, 1, 2, \ldots, n-1.$$

Example
Find the cube roots of 8.
First express 8 in polar form by locating the number 8 on an Argand diagram.

$$8 = 8(\cos 0^0 + i\sin 0^0)$$

Then the 3 cube roots are

$$z_k = 8^{1/3} = \left(\cos\frac{360k^0}{3} + i\sin\frac{360k^0}{3}\right), \quad k = 0, 1, 2.$$

They are

$$\begin{aligned} z_0 &= 2(\cos 0^0 + i\sin 0^0) = 2, \text{ for } k = 0 \\ z_1 &= 2(\cos 120^0 + i\sin 120^0) = 2\left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right] \\ &= -1 + \sqrt{3}i, \text{ for } k = 1 \\ z_2 &= 2(\cos 240^0 + i\sin 240^0) \\ &= 2\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = -1 - \sqrt{3}i, \text{ for } k = 2 \end{aligned}$$

Note: All the 3 cube roots of 8 lie on a circle of radius 2 in an Argand diagram as follows:

.

Example 2

Determine all the fifth roots of

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i.$$

First express $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$ in polar form

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i = (r, \theta)$$

where $r^2 = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1,\ r = 1$

$$\tan\theta = \frac{1/\sqrt{2}}{1/\sqrt{2}} = 1,\ \theta = 45^0$$

Then the fifth roots are

$$\begin{aligned} z_k &= 1^{1/5}\left(\cos\frac{45^0 + 360k^0}{5} + i\sin\frac{45^0 + 360k^0}{5}\right), \quad k = 0, 1, 2, 3, 4 \\ z_0 &= \cos\frac{45^0}{5} + i\sin\frac{45^0}{5} = \cos 9^0 + i\sin 9^0,\ \text{for } k = 0 \end{aligned}$$

$$\begin{aligned}
z_1 &= \cos 81^0 + i\sin 81^0, \text{ for } k = 1\\
z_2 &= \cos 153^0 + i\sin 153^0, \text{ for } k = 2\\
z_3 &= \cos 225^0 + i\sin 225^0, \text{ for } k = 3\\
z_4 &= \cos 297^0 + i\sin 297^0, \text{ for } k = 4
\end{aligned}$$

$\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^{1/5}$ on an Argand diagram

Practice Exercise 8E
Determine

(a) the square roots,

(b) the cube roots,

(c) the fourth roots, and

(d) the fifth roots of

(1)	1	(2)	-64
(3)	$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$	(4)	$16 - 16i$

Application of de-Moivre's theorem to trigonometric ratios.
Example 1
Express $\sin 5\theta$ in terms of
(a) mixed powers of $\sin\theta$ and $\cos\theta$

(b) powers of $\sin\theta$ only.
By de-Moivre's theorem and the binomial theorem,

$$\begin{aligned}
\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta \\
&\quad +10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta \\
&= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) \\
&\quad +i(5\cos^4\theta - \sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)
\end{aligned}$$

Equate the imaginary parts on both sides.

(a) $\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

(b) Since $\cos^2\theta = 1 - \sin^2\theta$, we have

$$\begin{aligned}
\sin 5\theta &= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 5\sin\theta - 20\sin^3\theta + 16\sin^5\theta.
\end{aligned}$$

Example 2
Express $\sin^5\theta$ in terms of sine and cosine of multiple angles.
If $z = \cos\theta + i\sin\theta$, then by de-Moivre's theorem:

$$\begin{aligned}
z^m &= \cos m\theta + i\sin m\theta && (1) \\
z^{-m} &= \cos(-m\theta) + i\sin(-m\theta) \\
z^{-m} &= \cos m\theta - i\sin m\theta && (2)
\end{aligned}$$

$\underline{(1)+(2)}$: $\quad z^m + z^{-m} = 2\cos m\theta \qquad (3)$

$\underline{(1)-(2)}$: $\quad z^m - z^{-m} = 2i\sin m\theta \qquad (4)$

Now by the binomial theorem and (4) above

$$\begin{aligned}
(2i\sin\theta)^5 &= (z - z^{-1})^5 \\
&= z^5 - 5z^4z^{-1} + 10z^3z^{-2} - 10z^2z^{-3} + 5zz^{-4} - z^{-5} \\
&= z^5 - 5z^3 + 10z - 10z^{-1} + 5z^{-3} - z^{-5}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow 32i^5 \sin^5\theta &= (z^5 - z^{-5}) - 5(z^3 - z^{-3}) + 10(z - z^{-1}) \\
\Rightarrow 32i \sin^5\theta &= 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta \text{ (using (4) above)} \\
\Rightarrow 32\sin^5\theta &= 2\sin 5\theta - 10\sin 3\theta + 20\sin\theta \\
\Rightarrow \sin^5\theta &= \frac{1}{32}(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta) \\
&= \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 20\sin\theta)
\end{aligned}$$

Practice Exercise 8F

1. Express (a) $\cos 3\theta$ (b) $\cos 6\theta$ in terms of
   (i) mixed powers of $\sin\theta$ and $\cos\theta$
   (ii) powers of $\cos\theta$ only
2. Express (a) $\sin 3\theta$ (b) $\sin 6\theta$ in terms of
   (i) mixed powers of $\sin\theta$ and $\cos\theta$
   (ii) powers of $\sin\theta$ only
3. Express in terms of multiple angles
   (a) $\sin^3\theta$ (b) $\cos^3\theta$ (c) $\sin^4\theta$
   (d) $\cos^4\theta$ (e) $\sin^6\theta$ (f) $\cos^6\theta$

Summary
We are able to perform the basic arithmetic operations on complex numbers in the Cartesian and polar forms and solve some polynomial equations with real number coefficients.
We then give several applications of de-Moivre's theorem on powers and roots of complex numbers, and also on trigonometric ratios.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 199-218.

LECTURE NINE

Algebra of Matrices

Introduction

Matrices are a generalization of a vector. A matrix is a rectangular arrangement (or array) of numbers into rows and columns, enclosed within curved or square brackets. A matrix consisting of m rows and n columns is called a $(m \times n)$-matrix.

Examples and Types

1. $(a_1 a_2 \ldots a_n)$ is a $(1 \times n)$-matrix called a row matrix or a row vector.
2. $\begin{pmatrix} a_1 \\ \vdots \\ a_m \end{pmatrix}$ is an $(m \times 1)$-matrix, called a column matrix or column vector.
3. An $(n \times n)$-matrix is called a square matrix of order n, with equal number of rows and columns. For example $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a square matrix of order 2.

4. $\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ is a (2×3)-matrix.

5. The transpose A^T of a matrix A is obtained from A by interchanging the rows and columns. For example

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

and $(a_1 \dots a_m)^T$ is an $(m \times 1)$-matrix.

6. If all the numbers in an $(m \times n)$-matrix A are zero, then A is called a zero matrix of order $m \times n$, and we write $A = 0$.

7. A diagonal matrix of order n is a square matrix of order n such that all the entries outside the leading diagonal are all zero. For example $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$ is a diagonal matrix of order 3.

8. A diagonal matrix in which all the numbers in the leading diagonal are equal to 1, is called an identity matrix I_n of order n. For example

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notation

1. Use capital letters of the alphabet to denote matrices, and lower case letters to denote the entries.

2. The entry a_{ij} is the entry located in the i-th row and j-th column. An $(m \times n)$-matrix A with entries a_{ij} is written in a shorthand form as

$$\begin{aligned} A &= (a_{ij})_{m,n} \quad \text{or} \quad A = [a_{ij}]_{m,n} \\ \text{or} \quad A &= (a_{ij})_n, \quad \text{if} \quad m = n. \end{aligned}$$

If $A = (a_{ij})_{m,n}$, then $A^T = (a_{ji})_{n,m}$.

Equality of Matrices
Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal $A = B$, if they are of the same order (i.e. they have equal number of rows and equal number of columns), and if all the corresponding entries are equal, i.e. $a_{ij} = b_{ij}$ for all (i, j).

Objectives
The reader should be able to

- add or subtract matrices of the same order;
- multiply a matrix by a scalar;
- multiply matrices which are compatible; and
- find the inverse of a non-singular (2×2)-matrix.

Pre-Test

1. If $A = \begin{bmatrix} 6 & 0 \\ -3 & 4 \\ 9 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 1 \\ 5 & -4 \\ 10 & 7 \end{bmatrix}$, find
   (a) $A + B$, (b) $A - B$ (c) $2B$ (d) $4A - 3B$

2. Evaluate $A \cdot B$ and $B \cdot A$ if they exist

   (a) $A = \begin{pmatrix} 1 & 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 & 6 \end{pmatrix}^T$

   (b) $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 7 \end{pmatrix}$

   (c) $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 \\ 6 & 7 \end{pmatrix}$

   (d) $A = \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$

   (e) $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$, $B = I_3$

3. Find the values of x and y so that

$$\begin{bmatrix} 2 & 7 \\ -3 & 0 \end{bmatrix} \cdot \begin{bmatrix} x & y \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 11 & 29 \\ -6 & 9 \end{bmatrix}$$

4. Calculate $f(A)$ if

$$A = \left(\begin{matrix} -3 & 0 \\ 1 & 5 \end{matrix} \right] \quad \text{and} \quad f(x) = 5x^2 - 7x + 8$$

5. Find the inverse, if it exists
   (a) $A = \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix}$ (b) $B = \begin{pmatrix} 6 & 8 \\ 3 & 4 \end{pmatrix}$

Addition of Matrices
Add matrices of the same order by adding corresponding entries. If $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{m,n}$, define

$$A + B = (a_{ij} + b_{ij})_{m,n}.$$

Example 1

$$\begin{aligned} \begin{pmatrix} 4 & 3 & 1 \\ -1 & 2 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 3 & 7 \\ -2 & -1 & 9 \end{pmatrix} &= \begin{pmatrix} 4+1 & 3+3 & 1+7 \\ -1-2 & 2-1 & 3+9 \end{pmatrix} \\ &= \begin{pmatrix} 5 & 6 & 8 \\ -3 & 1 & 12 \end{pmatrix} \end{aligned}$$

Example 2
If $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, then
$A + O = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = A$ and $O + A = A$.

Subtraction of Matrices
Subtract matrices of the same order by subtracting corresponding entries. If $A = (a_{ij})_{m,n}$ and $B = (b_{ij})_{m,n}$, define $A - B = (a_{ij} - b_{ij})_{m,n}$.

Example 3

$$\begin{pmatrix} 1 & 2 \\ 6 & 4 \\ -2 & 7 \end{pmatrix} - \begin{pmatrix} 1 & 6 \\ -2 & 4 \\ 7 & 5 \end{pmatrix} = \begin{pmatrix} 1-1 & 2-6 \\ 6+2 & 4-4 \\ -2-7 & 7-5 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 8 & 0 \\ -9 & 2 \end{pmatrix}$$

Multiplication by a Scalar

If $A = (a_{ij})_{m,n}$ and k is a number (called a scalar), define multiplication of A by k as a matrix whose entries are equal to k times the corresponding entries of A, and denote it by kA, i.e.

$$kA = k(a_{ij})_{m,n} = (ka_{ij})_{m,n}$$

Example 4

$$\begin{aligned} 3\begin{pmatrix} 2 & 4 & 5 \\ -1 & 0 & -3 \end{pmatrix} &= \begin{pmatrix} 3\times 2 & 3\times 4 & 3\times 5 \\ 3\times -1 & 3\times 0 & 3\times -3 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 12 & 15 \\ -3 & 0 & -9 \end{pmatrix} \end{aligned}$$

Practice Exercise 9A

If $A = \begin{pmatrix} 2 & -1 \\ 3 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 1 \\ 6 & 9 \end{pmatrix}$ find

(1) $2A + 3B$ (2) $-4A + 5B$

Multiplication of Matrices

If A and B are matrices, define the product $A \cdot B$ of A and B only if the number of columns of A is equal to the number of rows of B as follows:

If $A = (a_{ij})_{p,q}$ and $B = (b_{ij})_{q,r}$, then the i-th row of A is $\begin{pmatrix} a_{i1}a_{i2}\cdots a_{iq} \end{pmatrix}$ are the j-th column of B is $\begin{pmatrix} b_{1j}b_{2j}\cdots a_{qj} \end{pmatrix}^T$. If $AB = (c_{ij})_{p,r}$, then define

$$\begin{aligned} c_{ij} &= (a_{i1}a_{i2}\cdots a_{iq})(b_{1j}b_{2j}\cdots b_{qj})^T \\ &= a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{iq}b_{qj} \end{aligned}$$

(i.e. multiply the i-th row of A by the j-th column of B using the dot product in vectors).

Example 1
Determine AB and BA if they exist, if

$$A = \begin{pmatrix} 3 & 7 & -10 \\ 4 & 5 & 8 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 3 \\ 0 & 8 \end{pmatrix}$$

No. of columns of $A = 3 \neq 2 =$ No. of rows of B.
Therefore AB is not defined.
No. of columns of $B = 2 =$ No. of rows of A.
Therefore BA is defined, and

$$\begin{aligned} BA &= \begin{pmatrix} 2 & 3 \\ 0 & 8 \end{pmatrix} \cdot \begin{pmatrix} 3 & 7 & -10 \\ 4 & 5 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 2\times 3 + 3\times 4 & 2\times 7 + 3\times 5 & 2\times -10 + 3\times 8 \\ 0\times 3 + 8\times 4 & 0\times 7 + 8\times 5 & 0\times -10 + 8\times 8 \end{pmatrix} \\ &= \begin{pmatrix} 18 & 29 & 4 \\ 32 & 40 & 64 \end{pmatrix} \end{aligned}$$

Remarks

1. Since AB is not defined, we say that A and B are not compatible for multiplication.

2. Since BA is defined, we say that B and A are compatible for multiplication.

3. The order in which we multiply two matrices is, therefore, very important as BA may be defined but AB may not be defined.

Example 2
Find $A \cdot I_3$ and $I_3 \cdot A$, if they exist, if

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

No. of columns of $A = 3 =$ No. of rows of I_3.
Therefore $A \cdot I_3$ is defined and

$$A \cdot I_3 = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = A$$

No. of columns of $I_3 = 3 \neq 2 =$ No. of rows of A.
Therefore $I_3 \cdot A$ is not defined.

Remark
Whenever $A \cdot I_3$ is defined, $A \cdot I_3 = A$.
Therefore I_3 behaves like 1 for numbers under multiplication.

Example 3
Find a and b, given that

$$\begin{pmatrix} 4 & 2a & 3 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ 3 & 0 \\ -6 & 3b \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$$

$$\begin{pmatrix} 4 + 6a - 18 & 20 + 9b \\ -3 + 6 + 0 & -5 + 0 + 0 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ 5 & -5 \end{pmatrix}$$

$$\begin{aligned} 6a - 14 = -2 &\Rightarrow a = 2 \\ 20 + 9b = 2 &\Rightarrow b = -2 \end{aligned}$$

Example 4
Calculate $f(A)$ if

$$\begin{aligned} A &= \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \text{ and } f(x) = 1 + 3x + x^2 \\ f(A) &= I_2 + 3A + A^2 \\ A^2 &= \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
f(A) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 4\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 5 & 5 \\ 5 & 10 \end{pmatrix} \\
&= \begin{pmatrix} 1+6+5 & 0+3+5 \\ 0+3+5 & 1+9+10 \end{pmatrix} = \begin{pmatrix} 12 & 8 \\ 8 & 20 \end{pmatrix}
\end{aligned}
$$

Practice Exercise 9B

1. Calculate AB and BA if they exist.
   Compare to decide whether $AB = BA$ or not.

   (a) $A = \begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 8 \\ 2 & 1 \\ 3 & 7 \end{pmatrix}$

   (b) $A = \begin{bmatrix} -5 & 1 \\ 4 & 9 \end{bmatrix}$, $B = \begin{pmatrix} 8 & 1 \\ -2 & 3 \end{pmatrix}$

   (c) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Simplify
$$\begin{pmatrix} -2 & 5 & 7 \\ 3 & 1 & -4 \\ 6 & -3 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

3. Calculate $f(A)$ if
$$A = \begin{pmatrix} 1 & 2 \\ 7 & 5 \end{pmatrix} \quad \text{and} \quad f(x) = 5x^2 - 7x + 8.$$

4. Find the values of p and q so that
$$\begin{pmatrix} p & 6 \\ 5 & q \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

Inverse of a non-singular (2×2)-matrix
In order for $A \cdot B$ and $B \cdot A$ to be defined and be of the same order, we must consider square matrices of the same order.

Let A and B be square matrices of order n. If

$$A \cdot B = I_n \text{ and } B \cdot A = I_n$$

where I_n is the identity matrix of order n, we say that A and B are non-singular matrices. B is called the inverse of A, denoted by A^{-1} and A is called the inverse of B, denoted by B^{-1}.
i.e. $A^{-1} = B$ and $B^{-1} = A$.

Example 1.
Calculate, if $ad - bc \neq 0$

(a) $\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(a) $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \dfrac{1}{ad-bc}\begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$

Remarks

1. From Example 1 above,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

provided $ad - bc \neq 0$.

2. The number $ad-bc$ is therefore very important. It is used to determine when a (2×2)-matrix has an inverse.

3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the number $ad-bc$ is called the determinant of A i.e.

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

4. If $|A| \neq 0$, the matrix A is said to be non-singular or invertible.

5. If $|A| = 0$, the matrix A is said to be singular or non-invertible, i.e. A has no inverse.

Example 2
Find the inverse, if it exists.
(a) $A = \begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}$ (b) $A = \begin{bmatrix} 3 & 6 \\ 2 & 4 \end{bmatrix}$

$$\begin{aligned} \text{(a)} \quad |A| &= \begin{vmatrix} 3 & -5 \\ -2 & 4 \end{vmatrix} \\ &= 3(4) - (-5)(-2) \\ &= 12 - 10 = 2 \neq 0 \end{aligned}$$

Therefore A^{-1} exists.

$$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$
$$a = 3,\ b = -5,\ c = -2,\ d = 4$$
$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & \frac{5}{2} \\ 1 & \frac{3}{2} \end{pmatrix}$$

Check. $\begin{pmatrix} 3 & -5 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 2 & \frac{5}{2} \\ & \\ 1 & \frac{3}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(b) $|B| = \begin{vmatrix} 3 & 6 \\ 2 & 4 \end{vmatrix} = 12 - 12 = 0.$

Therefore B^{-1} does not exist.

Practice Exercise 9C

Find the inverse, if it exists

1. $A = \begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix}$ 2. $B = \begin{pmatrix} 15 & 10 \\ 6 & 4 \end{pmatrix}$

3. $C = \begin{pmatrix} 2 & 1 \\ 7 & 5 \end{pmatrix}$ 4. $D = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$

Summary

We are able to perform addition and subtraction of matrices of the same order, multiplication of a matrix by a scalar, and multiplication of matrices which are compatible. We are also able to find the inverse of any non-singular (2×2)-matrix.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 219-243.

LECTURE TEN

Rank of a Matrix

Introduction

We consider the three elementary operations on matrices, which are used to reduce any $(m \times n)$-matrix to row-reduced echelon form. The matrix and its row echelon form are row equivalent in the sense that they have the same rank. The rank of the echelon form is the number of its non-zero rows.
A square matrix whose rank equals the number of rows (or columns) is called a non-singular matrix. Otherwise it is called a singular matrix.

Objectives

The reader should be able to

- reduce any $(m \times n)$-matrix to its row-reduced echelon form.
- determine the rank of any $(m \times n)$-matrix
- apply the rank of a matrix to determine whether a square matrix is non-singular or singular.

Pre-Test

1. Reduce each matrix to its row-reduced echelon form.

(a) $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 1 & -4 \\ 2 & 3 & 2 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$

(c) $C = \begin{pmatrix} 1 & 2 & 8 & 1 \\ 1 & -3 & -7 & 6 \\ -1 & 1 & 1 & -4 \end{pmatrix}$

(d) $D = \begin{pmatrix} 5 & 2 & -3 \\ 1 & 3 & 4 \\ 7 & 5 & 9 \\ 0 & 1 & 1 \end{pmatrix}$

2. Find the rank of each matrix

(a) $A = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 1 & 3 & 1 & 6 \\ 1 & 5 & -1 & 16 \\ 4 & 1 & 0 & 2 \end{bmatrix}$

(b) $C = \begin{bmatrix} 8 & -1 & 6 & 3 & 7 \\ 3 & 0 & -1 & 4 & 3 \\ -1 & -1 & 9 & -9 & -2 \end{bmatrix}$

3. Determine whether each matrix is non-singular or singular, by finding the rank.

(a)

(c) $A = \begin{pmatrix} 7 & 6 & 5 \\ 1 & 2 & 1 \\ 3 & -2 & 1 \end{pmatrix}$

(b) $B = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & 1 \\ 1 & -2 & 0 \end{pmatrix}$

Elementary Operations on Matrices
Consider the following system of equations:

$$\begin{aligned} x+2y+8x+w &= 1 \\ x-3y-7z+6w &= 2 \\ -x+y+z-4w &= 3 \end{aligned}$$

The system can be written in matrix form as:

$$\begin{pmatrix} 1 & 2 & 8 & 1 \\ 1 & -3 & -7 & 6 \\ -1 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

To solve the system, we use the following 3 elementary row operations on the augmented matrix

$$\left(\begin{array}{rrrr|r} 1 & 2 & 8 & 1 & 1 \\ 1 & -3 & -7 & 6 & 2 \\ -1 & 1 & 1 & -4 & 3 \end{array}\right)$$

I. Interchanging the i-th and j-th rows, denoted by R_{ij}

II. Multiplying the i-th row by a non-zero scalar k, denoted by $k \cdot R_i$ or $R_i(k)$

III. Adding a multiple of the i-th row to the j-th row, denoted by $R_j + kR_i$ or $R_{ji}(k)$.

Row-reduced echelon matrix
An $m \times n$ matrix is called a **row-reduced echelon matrix** if it has the following properties:

(i) The leading entry of any non-zero row is 1.

(ii) All other entries below in the column containing the leading entry are zeroes.

(iii) The leading entry in the i-th row is strictly to the left of the leading entry in the $(i+1)$-th row.

(iv) All the zero rows, if any, are placed at the last rows.

Remark.
It follows that in a row-reduced echelon form of a matrix, each row has more leading zeroes than the row preceding it.

Example 1.
The following are row-reduced echelon matrices

(a) $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Example 2
Use elementary row operations to reduce the following matrix to its row-reduced form.

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 2 \\ 2 & 1 & -4 \end{pmatrix}$$

Solution

$$\begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 2 \\ 2 & 1 & -4 \end{pmatrix} \xrightarrow{R_{13}} \begin{pmatrix} 2 & 1 & -4 \\ 2 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_1} \begin{pmatrix} 1 & \frac{1}{2} & -2 \\ 2 & 3 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_2-2R_1} \begin{pmatrix} 1 & \frac{1}{2} & -2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{pmatrix} \xrightarrow{\frac{1}{2}R_2} \begin{pmatrix} 1 & \frac{1}{2} & -2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3-R_2} \begin{pmatrix} 1 & \frac{1}{2} & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Definition.

1. The **rank** of a matrix is the number of non-zero rows in its row-reduced echelon form. In Example 2 above, the rank is 2.

2. If the rank of a square $n \times n$ matrix A is equal to n, then we say that A is a **non-singular** matrix. If the rank is less than n, say A is a **singular** matrix. The matrix in Example 2 is a singular matrix.

Practice Exercise 10

Find the rank of each matrix. If the matrix is $n \times n$, determine if it is singular or non-singular.

(1) $\begin{pmatrix} 1 & 2 & 8 & 1 \\ 1 & -3 & -7 & 6 \\ -1 & 1 & 1 & -4 \end{pmatrix}$ (2) $\begin{pmatrix} 3 & 1 & 2 \\ 2 & 1 & 2 \\ 6 & 2 & 5 \end{pmatrix}$

Summary

We use elementary operations to find the rank of any $(m \times n)$-matrix by reducing it to its echelon form.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000, pages 247-252.

LECTURE ELEVEN

Systems of Linear Equations

Introduction

We shall use elementary row operations to solve any system of m linear equations in n unknowns.

Objective
The reader should be able to solve any system of m linear equations in n unknowns, using elementary operations.

Pre-Test
Find all the solutions of the following systems of linear equations

1. $$\begin{aligned} 3x + y + 2z &= 1 \\ 2x + y + 2z &= 2 \\ 6x + 2y + 5z &= 3 \end{aligned}$$

2. $$\begin{aligned} x - 3y + 2z &= 0 \\ -2x + y - 3z &= 0 \end{aligned}$$

3. $$\begin{aligned} x + y + z &= 1 \\ 2x + z &= -2 \\ 2y + z &= 0 \end{aligned}$$

4. $$\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + z &= 0 \\ 3x + 2y + z &= -3 \end{aligned}$$

Types of Solutions
A system of linear equations may have

(a) only one solution

(b) no solution

(c) more than one solutions.

Systems of linear equations, which have one or more than one solution, are said to be **consistent** or **solvable**. Those which have no solution are said to be **inconsistent**.

Systems with only one solution
These are systems where

(i) the number of equations is equal to the number of unknowns, and

(ii) the square matrix of coefficients is non-singular.

Example 1. Solve

$$\begin{aligned} 2x + y - z &= 1 \\ 2y + z &= 2 \\ 5x + 2y - 3z &= 3 \end{aligned}$$

Solution
Write the system in matrix form as

$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Row-reduce the augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 1 & -1 & 1 \\ 0 & 2 & 1 & 2 \\ 5 & 2 & -3 & 3 \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 & 2 \\ 5 & 2 & -3 & 3 \end{array}\right)$$

$$\xrightarrow{R_3-5R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 1 & 2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right) \xrightarrow{\frac{1}{2}R_1} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\xrightarrow{R_3+\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & -\frac{1}{4} & 1 \end{array}\right) \xrightarrow{-4R_3} \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & 1 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

The system becomes

$$x + \frac{1}{2}y - \frac{1}{2}z = \frac{1}{2} \qquad (1)$$
$$y + \frac{1}{2}z = 1 \qquad (2)$$
$$z = -4 \qquad (3)$$

Use (3) in (2):

$$y + \frac{1}{2}(-4) = 1, \quad y = 3$$

(1) becomes $x + \frac{1}{2}(3) - \frac{1}{2}(-4) = \frac{1}{2}$

$$x = -3$$

The only solution is $(x, y, z) = (-3, 3, -4)$.
Note. In Example 1 above, the rank of the matrix of coefficients is 3 and so it is non-singular.

Systems with more than one solution
These are systems where

(i) the rank of the matrix of coefficients is less than the number of unknowns, and

(ii) the ranks of the matrix of coefficients and the augmented matrix are equal.

Example 2. Solve

$$\begin{aligned} x + 2y + 8z &= 1 \\ x - 3y - 7z &= 6 \\ -x + y + z &= -4 \end{aligned}$$

Solution
First find the ranks of the matrix of coefficients and the augmented matrix

$$\left(\begin{array}{rrr|r} 1 & 2 & 8 & 1 \\ 1 & -3 & -7 & 6 \\ -1 & 1 & 1 & -4 \end{array}\right) \underset{R_3 + R_1}{\overset{R_2 - R_1}{\longrightarrow}} \left(\begin{array}{rrr|r} 1 & 2 & 8 & 1 \\ 0 & -5 & -15 & 5 \\ 0 & 3 & 9 & -3 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{5}R_2} \left(\begin{array}{rrr|r} 1 & 2 & 8 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 3 & 9 & -3 \end{array}\right) \xrightarrow{R_3 - 3R_2} \left(\begin{array}{rrr|r} 1 & 2 & 8 & 1 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Rank of matrix of coefficients = Rank of augmented matrix = 2.
The rank is less than the number of variables (unknowns). The system becomes

$$\begin{aligned} x + 2y + 8z &= 1 \qquad (1) \\ y + 3z &= -1 \qquad (2) \end{aligned}$$

From (2), $y = -3z - 1$
(1) becomes $x + 2(-3z - 1) + 8z = 14$.

$$x = -2z - 3$$

Solution is

$$(x, y, z) = (-2z - 3,\ -3z - 1,\ z)$$

If we put $z = \alpha$, an arbitrary constant,

$$(x, y, z) = (-2\alpha - 3,\ -3\alpha - 1,\ \alpha)$$

for any real number α, or

$$(x, y, z) = \alpha(-2, -3, 1) + (-3, -1, 0).$$

Systems with no solution
These are systems where the rank of the matrix of coefficients $\neq$ the rank of the augmented matrix.

Example 3
Determine whether the system is consistent or not.

$$\begin{aligned} x + 2y + 3z &= 3 \\ 2x + 3y + 4z &= 1 \\ 3x + 4y + 5z &= 2 \end{aligned}$$

Solution
Row-reduce the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 5 & 2 \end{array}\right) \begin{array}{c} R_2 - 2R_1 \\ \longrightarrow \\ R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & -1 & -2 & -5 \\ 0 & -2 & -4 & -7 \end{array}\right)$$

$$\xrightarrow{(-1)R_2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & -2 & -4 & -7 \end{array}\right) \xrightarrow{R_3+2R_2} \left(\begin{array}{cccc} 1 & 2 & 3 & 3 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

Rank of matrix of coefficient $= 2$.
Rank of augmented matrix $= 3$.
Therefore the system has no solution, i.e. is inconsistent.

Practice Exercise 11

Solve

1. $$\begin{aligned} x + 2y + z &= 2 \\ 3x + y - 2z &= 1 \\ 4x - 3y - z &= 3 \end{aligned}$$

2. $$\begin{aligned} 2x - 7z &= 0 \\ y - 3z &= 0 \end{aligned}$$

3. $$\begin{aligned} x + 3y - 2z &= 1 \\ -x + 4y + z &= 2 \\ 5x - 6y - 7z &= 3 \end{aligned}$$

4. $$\begin{aligned} 2x + 3y - 2z &= -5 \\ -x + 4y + 2z &= 0 \\ 5x - 6y - 7z &= -5 \end{aligned}$$

5. $$\begin{aligned} x + y - 2z &= 3 \\ 3x - y - 5z &= 2 \\ -2x + 2y + 3z &= 1 \end{aligned}$$

6. $$\begin{aligned} x + 2y + z &= 2 \\ 3x + y - 2z &= 1 \\ 4x + 3y - z &= 4 \end{aligned}$$

Summary

We use elementary operations on the matrix of coefficients and the augmented matrix to solve any system of m linear equations in n unknowns; consisting of those with only one solution, those with no solution and those with more than one solutions.

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000. Pages 252-266.

LECTURE TWELVE

Determinants

Introduction

In Lecture 9, we define the determinant of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

as

$$|A| = \det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

and

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

It follows that the inverse of A exists only if $ad - bc \neq 0$. Hence the determinant determines when a quare matrix has an inverse.
In this Lecture, we shall consider the determinant of a 3×3 matrix.

Objective

The reader should be able to evaluate the determinant of a 3×3 matrix.

Pre-Test
Find the determinant of

1. $\begin{pmatrix} 1 & -2 & -3 \\ 3 & 3 & 2 \\ 2 & 3 & -1 \end{pmatrix}$ 2. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 7 & -2 & 1 \end{pmatrix}$

Solve the equations

3. $\begin{vmatrix} x+1 & 1 \\ -1 & x-1 \end{vmatrix} = 0.$ 4. $\begin{vmatrix} x-1 & 3 & 1 \\ x & -2 & 1 \\ 2 & 1 & x \end{vmatrix} = 0$

Determinant of a 3×3 matrix
We find the determinant of a 3×3 matrix by using the formula for the determinant of a 2×2 matrix. We can use any row or column to expand the determinant of a 3×3 matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

together with the following sign rule:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

There are, therefore, 6 different ways of finding the determinant.
If we use the first row to expand we have

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

If we use the 2nd column

$$|A| = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

Check that the two expansions are equal.

Diagonal Method
There is a special method for finding $|A|$.

$$\begin{aligned}
&\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
= \;& a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{21}a_{32}a_{13} \\
& -(a_{13}a_{22}a_{31} + a_{12}a_{21}a_{23} + (a_{23}a_{32}a_{11})
\end{aligned}$$

Example 1. Evaluate

$$\begin{vmatrix} 2 & 3 & -4 \\ 1 & 5 & -2 \\ 3 & -1 & 6 \end{vmatrix}$$

Solution
Using the diagonal method

$$\begin{aligned}
\begin{vmatrix} 2 & 3 & -4 \\ 1 & 5 & -2 \\ 3 & -1 & 6 \end{vmatrix} & \begin{array}{l} = 2(5)6 + 3(-2)3 + (1)(-1)(-4) \\ -[3(5)(-4) + 1(3)6 + (-1)(-2)2] \end{array} \\
= \;& 60 - 18 + 4 - (-60 + 18 + 4) \\
= \;& 46 + 38 = 84
\end{aligned}$$

Example 2
Solve the equation

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & x & x^2 \\ 1 & 3 & 9 \end{vmatrix} = 9x + 2x^2 + 12 - (4x + 18 + 3x^2)$$

i.e.

$$\begin{aligned}
-x^2 + 5x - 6 &= 0 \\
x^2 - 5x + 6 &= 0 \\
(x-3)(x-2) &= 0 \\
x &= 2 \text{ or } 3
\end{aligned}$$

Example 3
Use the determinant to test if the following matrix is non-singular.

$$\begin{bmatrix} 5 & 3 & -1 \\ 1 & 1 & 4 \\ 2 & -3 & 6 \end{bmatrix}$$

Solution.
Find the determinant.

$$\begin{vmatrix} 5 & 3 & -1 \\ 1 & 1 & 4 \\ 2 & -3 & 6 \end{vmatrix} \begin{aligned} &= 30 + 24 + 3 - (-2 + 18 - 60) \\ &= 57 + 44 = 101 \end{aligned}$$

Since the determinant is not equal to zero, it follows that the matrix is non-singular.

Practice Exercise 12
Evaluate

1. $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$

2. $\begin{vmatrix} 7 & 3 & 5 \\ 3 & 1 & 4 \\ 5 & 4 & -2 \end{vmatrix}$

Solve the equations

3. $\begin{vmatrix} 1 & 6 & 5 \\ 1 & 3x & x+3 \\ 1 & 2x & x+2 \end{vmatrix} = 0$

4. $\begin{vmatrix} 1 & x & x^2 \\ 1 & 3 & 9 \\ 1 & 1 & 1 \end{vmatrix} = 0$

Test if the following matrices are non-singular, using the determinant

5. $\begin{vmatrix} 7 & 1 & -4 \\ 6 & -9 & -15 \\ -2 & 3 & 5 \end{vmatrix}$

6. $\begin{vmatrix} 5 & 2 & 5 \\ 1 & 4 & -1 \\ 3 & 7 & 3 \end{vmatrix}$

Summary
We consider seven different ways of expanding a 3×3 determinant. When the determinant is non-zero, then the matrix is non-singular.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000. Pages 273-287.

LECTURE THIRTEEN

Cramer's Rule

Introduction

We now consider a method called Cramer's Rule which uses only determinants to evaluate the only one solution of a system of 3 linear equations in 3 unknowns, when the matrix of coefficients has non-zero determinant, or is non-singular.

Objective

The reader should be able to use Cramer's Rule to solve systems of linear equations which have only one solution.

Pre-Test

Use Cramer's Rule to solve:

1. $$\begin{aligned} x - 2y - 3z &= 1 \\ 3x + 5y + 2z &= -1 \\ 2x + 3y - z &= 2 \end{aligned}$$

2. $$\begin{aligned} x + 2y + 3z &= 10 \\ 2x - 3y + z &= 1 \\ 3x + y - 2z &= 9 \end{aligned}$$

Cramer's Rule:

Given a system of linear equations

$$a_{11}x + a_{12}y + a_{13}z \;=\; b_1$$

$$\begin{aligned} a_{21}x + a_{22}y + a_{23}z &= b_2 \\ a_{31}x + a_{32}y + a_{33}z &= b_3 \end{aligned}$$

such that

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

If B_i represents the matrix obtained from the matrix of coefficients A by replacing the i-th column by $(b_1 \;\; b_2 \;\; b_3)^T$ for $i = 1, 2, 3$, then the unique solution of the system is

$$x = \frac{|B_1|}{|A|}, \quad y = \frac{|B_2|}{|A|} \quad \text{and} \quad z = \frac{|B_3|}{|A|}$$

Note: To use Cramer's rule, we only need to calculate 4 determinants which are $|A|, |B_1|$, $|B_2|$ and $|B_3|$.

Example. Use Cramer's Rule to solve:

$$\begin{aligned} 3x - 2y + 3z &= 8 \\ x + y - 2z &= -3 \\ 2x + y + 4z &= 16 \end{aligned}$$

Solution
The determinant of the matrix of coefficients is

$$\begin{vmatrix} 3 & -2 & 3 \\ 1 & 1 & -2 \\ 2 & 1 & 4 \end{vmatrix} \begin{aligned} &= 12 + 8 + 3 = (6 - 8 - 6) \\ &= 23 + 8 = 31 \neq 0 \end{aligned}$$

So we can use Cramer's Rule as follows:

$$\begin{aligned} x &= \frac{1}{31}\begin{vmatrix} 8 & -2 & 3 \\ -3 & 1 & -2 \\ 16 & 1 & 4 \end{vmatrix} = \frac{1}{31}[32 + 64 - 9 - (48 + 24 - 16)] \\ &= \frac{1}{31}(87 - 56) = \frac{31}{31} = 1 \end{aligned}$$

$$
\begin{aligned}
y &= \frac{1}{31}\begin{vmatrix} 3 & 8 & 3 \\ 1 & -3 & -2 \\ 2 & 16 & 4 \end{vmatrix} \\
&= \frac{1}{31}[-36 - 32 + 48 - (-18 + 32 - 96)] \\
&= \frac{1}{31}(-20 + 82) = \frac{62}{31} = 2 \\
z &= \frac{1}{31}\begin{vmatrix} 3 & -2 & 3 \\ 1 & 1 & -3 \\ 2 & 1 & 16 \end{vmatrix} \\
&= \frac{1}{31}[48 + 12 + 8 - (16 - 32 - 9)] \\
&= \frac{1}{31}(68 + 25) = \frac{93}{31} = 3
\end{aligned}
$$

i.e. $(x, y, z) = (1, 2, 3)$.

Practice Exercise

Use Cramer's Rule to solve:

1. $\begin{aligned} x + 2y + z &= 7 \\ 3x + y &= -2 \\ 5x + 5y + 2z &= 12 \end{aligned}$

2. $\begin{aligned} x + 3y - 4z &= 1 \\ -x + y - 3z &= 14 \\ y - 3z &= 5 \end{aligned}$

Summary

Cramer's Rule is a method which uses only determinants to solve a system of 3 linear equations in 3 unknowns when the determinant of the matrix of coefficients is non-zero

Post-Test

See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000. Pages 287-290.

LECTURE FOURTEEN

Translations in the Plane

Introduction

We shall consider translations of graphs and curves in the Plane, parallel to the x-axis and parallel to the y-axis.

Objectives

The reader should be able to:

- calculate the equation of a function whose graph is translated parallel to the axes in the plane; and
- describe the translations needed to transform the graph of a given function to another given function.

Pre-Test

1. If the graph of the function $y = f(x)$ is translated 2 units to the right along the x-axis and 3 units downwards along the y-axis, write down the equation of the new curve.
2. Given the graph of $x^2 + y^2 = a^2$, describe the translations needed to obtain the graph of

$$x^2 + y^2 - 8x - 6y = 0$$

3. Determine the equation of the new graph obtained by translation 2 units to the left along the x-axis and 1 unit upwards along the y-axis the graph of $x^2 + y^2 = 25$.

4. Describe in terms of the graph of $y = f(x)$, the graph of $y = f(x-3)+2$.

Translation along the y-axis

The function $y = f(x) + 3$ represents the graph of the function $y = f(x)$ translated 3 units upwards along the y-axis.
$y = f(x) + 3$ can be written as

$$y - 3 = f(x)$$

Similarly the function

$$y + 2 = f(x)$$

represents the graph of $y = f(x)$ translated 2 units downwards along the y-axis..

Translation along the x-axis

Using the same technique as for translation along the y-axis, we conclude that the function $y = f(x - 2)$ represents the graph of the function $y = f(x)$ translated 2 units to the right along the x-axis.

Similarly, $y = f(x + 3)$ represents the graph of $y = f(x)$ translated 3 units to the left along the x-axis.

Example 1

Describe in terms of the graph of $y = f(x)$, the graph of $y + 2 = f(x + 3)$.

Solution

It is the graph of $y = f(x)$ translated 2 units downward along the y-axis and 3 units to the left along the x-axis.

Example 2

Write down the equation of the new curve when the graph of $y = f(x)$ is translated 4 units to the right along the x-axis and 5 units downward along the y-axis.

Solution
The equation is
$$y + 5 = f(x - 4)$$
or
$$y = f(x - 4) - 5$$

Example 3
Given the graph of $x^2 + y^2 = a^2$, describe the translations needed to obtain the graph of
$$x^2 + y^2 - 2x + 4y = 20$$

Solution
Use the method of completing the square;

$$\begin{aligned}
& x^2 + y^2 - 2x + 4y = 20 \\
\Rightarrow \quad & (x^2 - 2x) + (y^2 + 4y) = 20 \\
\Rightarrow \quad & (x^2 - 2x + 1^2) + (y^2 + 4y + 2^2) = 20 + 1^2 + 2^2 \\
\Rightarrow \quad & (x - 1)^2 + (y + 2)^2 = 25 = 5^2
\end{aligned}$$

The translations needed are a translation by 1 unit to the right along the x-axis and a translation by 2 units downwards along the y-axis, where $a = 5$.

Example 4
Determine the equation of the new graph obtained by translation 3 units to the right along the x-axis and 4 units downwards along the y-axis the graph of
$$9x^2 + 4y^2 = 36$$

Solution
The equation is
$$9(x - 3)^2 + 4(y + 4)^2 = 36$$
or
$$9x^2 + 4y^2 - 54x + 32y + 109 = 0.$$

Practice Exercise 14

1. If the graph of the function $y = f(x)$ is translated 3 units to the left along the x-axis and 2 units upwards along the y-axis, write down the equation of the new curve.

2. Given the graph f $x^2 + y^2 = a^2$, describe the translations needed to obtain the graph of

$$x^2 + y^2 + 4x - 6y - 12 = 0$$

3. Determine the equation of the new graph obtained by translation 5 units to the right along the x-axis and 2 units upwards along the y-axis, the graph of $y^2 = 8x$.

4. Describe in terms of the graph of $y = f(x)$, the graph of $y = f(x+4)-3$.

Summary
We show that the graph of $y - 5 = f(x + 6)$ is the graph of $y = f(x)$ translated 6 units to the left along the x-axis and 5 units upwards along the y-axis.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000. Pages 292-295.

LECTURE FIFTEEN

Sets

Introduction

We shall examine how to describe sets, state different types of sets, and consider binary operations on sets, including modulo arithmetic on natural numbers.

Objectives

The reader should be able to:

- describe a set;
- state different types of sets;
- manipulate binary operations on sets; and
- apply the properties of a binary operation in modulo arithmetic on natural numbers.

Pre-Test

1. Write, by listing the members, the set of months in the year, which begin with J.

2. Obtain all the subsets of the set $\{P, D, S\}$.

3. Let A be a set in a universal set $\mathcal{U}$, such that A^c is the complement of A.
   Draw the operation table for

$$(\{\phi, A, A^c, \mathcal{U}\}, \cup)$$

   where $\cup$ is the operation of union.

4(a) If $\mathbb{N}$ is the set of all natural numbers, is $(\mathbb{N}, -)$ closed?

(b) If $\mathbb{Z}$ is the set of all integers, is $(\mathbb{Z}, -)$, (i) associative?, (ii) commutative?

(c) If $\mathbb{R}^*$ is the set of all non-zero real numbers, is $(\mathbb{R}^*, \div)$

   (i) associative?
   (ii) commutative?

   [For each of (a), (b), (c), give reason/counter-example].

5. Consider $(\{2, 4, 6, 8\}, \otimes)$, where $a \otimes b$ is the remainder when ab is divided by 10. Determine, from the operation table, if it exists (a) an identity element; (b) inverse of each element.

What is a set?
A **set** is a collection of well-defined objects, so that it is possible to decide whether any given object is, or is not, in the set.
An object in a set is called an **element** or a **member** of the set.

Notation and Representation
Use capital letters $A, B, C, D, \ldots$ to represent sets and lower case letters $a, b, c, d, \ldots$ to represent elements of a set..
'$a \in A$' denotes 'a is a member or an element of set A'
'$a \notin A$' denotes 'a is not a member or an element of set A'.

Description of sets

(i) **Using a diagram.** A set can be described by drawing the objects in the set in a diagram.

(ii) **By Word Description.** A set can be described in words. For example a set containing the numbers 1, 2, 3, 4, 5 can be described as '*a* set of natural numbers from 1 to 5.

(iii) **By Listing the Elements.** A set can be described by listing the elements inside a pair of braces or curly brackets $\{\cdots\}$. For example, a set consisting of numbers 1,2,3,4,5 is described as

$$\{1, 2, 3, 4, 5\}$$

(iv) **Sample Elements Description.** A set can be described by using a sample element to describe the set. For example, the set consisting of all odd positive integers can be described as

$$\{x : x \text{ is an odd positive integer}\}$$

or

$$\{x|x \text{ is an odd positive integer}\}$$

where : or | is read as 'such that'.

Example 1
If $\mathcal{U} = \{2, 4, 6, 8, \ldots, 20\}$ is the universal set, write down by listing the elements, the set {multiples of 3}.

Solution

$$\{\text{multiples of } 3\} = \{6, 12, 18\}.$$

Different Types of Sets

1. **Null or Empty Set.** The set containing no element is called the null set or the empty set. It is denoted by ϕ or { }.

2. **Singleton set.** A set containing only one member is called a singleton set.

3. **Finite and Infinite sets.** If the number of elements in a set is finite, the set is called a finite set. The number of elements in a finite set A is denoted by $n(A)$.
   A set containing an infinite number of elements is called an infinite set. For example, the following sets are infinite sets:

   (a) $\{x : x \text{ is a natural number}\} = \{1, 2, 3, 4, 5, \ldots\}$

   (b) $\{x | x \text{ is an integer}\} = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

4. **Subsets.** A set B, consisting of some or all the elements of a set A, is called a subset of A. '$B \subseteq A$' denotes 'B is a subset of A'.
   Every set is a subset of itself, i.e. $A \subseteq A$ for any set A. The empty set is considered a subset of every set, i.e. $\phi \subseteq A$, for any set A. If A is a finite set containing n elements, then there are 2^n subsets that can be formed from set A.

5. **Equal sets.** Two sets A and B are equal (denoted $A = B$), if they contain identically the same elements (not necessarily in the same order). Thus $A = B$ means
$$A \subseteq B \text{ and } B \subseteq A.$$

6. **Universal Sets.** The set containing all elements under discussion in a particular problem, is called the universal set for that problem. The universal set is denoted by $\mathcal{U}$ or ϵ. Then all other sets in the problem are subsets of this universal set. Thus the universal set changes from problem to problem.

7. **Complement.** If A is any set in a universal set $\mathcal{U}$, then the complement of A, denoted by A^c or A', is the set consisting of all the elements in $\mathcal{U}$ which are not in A.

8. **Venn Diagrams.** A Venn diagram consists of a rectangle, which represents the universal set in a particular problem, and circles inside the rectangle, representing subsets of the universal set.

Example 2
Obtain all the subsets of the set $\{a, b, c, d\}$.

Solution
There are $2^4 = 16$ subsets.
Subset with 0 element (1): ϕ
Subsets with 1 element (4): $\{a\}, \{b\}, \{c\}, \{d\}$
Subsets with 2 elements (6): $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$
Subsets with 3 elements (4):
$\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$
Subsets with 4 elements (1): $\{a, b, c, d\}$
Total $= 1 + 4 + 6 + 4 + 1 = 16$.

Operations on Sets
Union of Sets. If A and B are sets, define the union of A and B, denoted by $A \cup B$, as the set

$$A \cup B = \{x : x \in A \text{ or } x \in B\}$$

If $A \subseteq B$, then $A \cup B = B$.

Intersection of Sets. If A and B are sets, define the intersection of A and B, denoted by $A \cap B$, as the set

$$A \cap B = \{x| : x \in A \text{ and } x \in B\}$$

Thus $A \cap B$ consists of common elements in A and B.
If $A \subseteq B$, then $A \cap B = A$.

Example 3
Let A be a set in a universal set $\mathcal{U}$. such that A^c is the complement of A. Draw the operation table for

$$(\{\phi, A, A^c, \mathcal{U}\}, \cap)$$

Solution

Operation Table

$\cap$	ϕ	A	A^c	$\mathcal{U}$
ϕ	ϕ	ϕ	ϕ	ϕ
A	ϕ	A	ϕ	A
A^c	ϕ	ϕ	A^c	A^c
$\mathcal{U}$	ϕ	A	A^c	$\mathcal{U}$

Properties of Binary Operations

1. **Closure.** Let S be a set. An operation $*$ on S is a **binary operation** if for every pair of elements a, b in S, $a * b$ is in S. Then say that set S is closed with respect to the binary operation $*$, or say that $*$ satisfies the closure property on set S.

2. **Commutative Property:** Let $(S, *)$ be a set S together with a binary operation $*$ on S is said to satisfy the commutative law or property, if for every pair a, b in S,
$$a * b = b * a$$

3. **Associative Property:** Let $(S, *)$ be a set S together with a binary operation $*$ on S. The binary operation $*$ on S is said to satisfy the associative law or property, if for every triple a, b, c in S,
$$(a * b) * c = a * (b * c)$$

4. **Identity Element:** Let $(S, *)$ be a set S together with a binary operation $*$ on S. If there is an element denoted generally by e, in S such that
$$e * a = a * e \text{ for all } a \in S$$
then e is called an identity element for $*$.

5. **Inverses:** Let $(S, *)$ be a set S together with a binary operation $*$ on S, having an identity e. If a and b are elements in S such that
$$a * b = b * a = e$$

then a is called the inverse of b, and b is called the inverse of a in $(S, *)$. Denote the inverse of a by a^{-1}. Thus

$$a * b = b * a = e \Rightarrow b = a^{-1} \text{ and } a = b^{-1}$$

6. **Distributive Laws:** Let $(S, *, \circ)$ be a set together with two binary operation $*$ and $\circ$ on S. If for every a, b, c in S

$$a * (b \circ c) = (a * b) \circ (a * c)$$

and

$$(b \circ c) * a = (b * a) \circ (c * a)$$

then we say that $*$ is distributive over $\circ$.

Notation for Sets of Numbers

$\mathbb{N}$ = the set of all counting or natural numbers
$\mathbb{I}$ or $\mathbb{Z}$ = the set of all integers
$\mathbb{Q}$ = the set of all rational numbers
$\mathbb{R}$ = the set of all real numbers
$\mathbb{C}$ = the set of all complex numbers.

Example 4

(a) If $\mathbb{N}$ is the set of all natural numbers, is $(\mathbb{N}, \div)$ closed? Give reason/counter example.

(b) If $\mathbb{R}$ is the set of all real numbers, is $(\mathbb{R}, -)$ (i) associative?, (ii) commutative?. [Give reason/counter-example].

Solution

(a) Counter-example:
$a = 3$, $b = 4$, $a \div b = \frac{3}{4} \notin \mathbb{N}$.
Therefore $(\mathbb{N}, \div)$ is not closed.

(b)(i) $a = 2$, $b = 3$, $c = 4$
$(a - b) - c = (2 - 3) - 4 = -1 - 4 = -5$
$a - (b - c) = 2 - (3 - 4) = 2 + 1 = 3$
Hence $(2 - 3) - 4 \neq 2 - (3 - 4)$
Therefore $(\mathbb{R}, -)$ is **NOT** associative.

(ii) $a = 5$, $b = 7$
$a - b = 5 - 7 = -2$
$b - a = 7 - 5 = 2$
Hence $5 - 7 \neq 7 - 5$
Therefore $(\mathbb{R}, -)$ is **NOT** commutative.

Remark: For a property not to hold, it is sufficient to give a counter-example.

Example 5
Consider the set $S = \{4, 8, 12, 16\}$ and a binary operation $\otimes$ on S defined by $a \otimes b$ is the remainder when ab is divided by 20 (i.e. multiplication modulo 20)

(a) Draw the Operation Table for $(S, \otimes)$.

(b) From the Operation Table, determine if it exists

(i) an identity

(ii) an inverse of each element in S.

Solution

(a) **Operation Table**

$\otimes$	4	8	12	16
4	16	12	8	4
8	12	4	16	8
12	8	16	4	12
16	4	8	12	16

(b)(i) From the last row and last column of the Operation Table, 16 is an identity.

(ii) From the Operation Table

$4 \otimes 4 = 16 \Rightarrow 4^{-1} = 4$
$8 \otimes 12 = 16 \Rightarrow 8^{-1} = 12, 12^{-1} = 8$
$16 \otimes 16 = 16 \Rightarrow 16^{-1} = 16.$

Example 6
If $\mathcal{P}(X)$ is the set of all subsets of a set X, then in $(\mathcal{P}(X), \cup, \cap)$,

(a) $\cup$ is distributive over $\cap$
i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $\cap$ is distributive over $\cup$
i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
$a(b + c) = ab + ac$, for all a, b, c in $\mathbb{R}$.

Example 7
Multiplication is distributive over addition.

Practice Exercise 15

1. Write the set $\{x| - 3 < x \leq 5,\ x \text{ is an integer}\}$ by listing the elements.

2. Write down all the subsets of the set $\{a, b, c\}$.

3. Given the universal set $\mathcal{U} = \{1, 2, 3, 4, 5\}$, $M = \{1, 2, 5\}$ and $N = \{1, 3, 5\}$, find (a) $(M \cup N)^c$, (b) M^c, (c) N^c, (d) $M^c \cap N^c$, (e) $M^c \cup N^c$.

4. In $(\mathbb{R}, *)$, where

$$x * y = x + y - xy \text{ for all } x, y \text{ in } \mathbb{R}$$

   (a) Is the operation $*$ associative?
   (b) Does $(\mathbb{R}, *)$ contain an identity?
   (c) What members of $\mathbb{R}$ have inverses?

5. If $S = \{1, 3, 5, 7\}$ and $*$ is multiplication modulo 8,

   (a) Draw the Operation Table for $(S, *)$;
   (b) From the Operation Table, determine if it exists,
      (i) an identity
      (ii) an inverse for each element in S.

Summary
We define a set and state several different types of sets. We then consider the operations of union and intersection of sets. We give properties of a general binary operation on a set and give examples which include modulo arithmetic on natural numbers.

Post-Test
See Pre-Test at the beginning of the Lecture.

Reference

S.A. Ilori and D.O.A. Ajayi. Algebra. University Mathematics Series (2), Y-Books, Ibadan, Nigeria, 2000. Pages 325-352.